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Volume 11 *No.* 42 *June* 1960

D. G. Northcott: Semi-regular rings and semi-regular ideals 81

A. H. Stone: Universal spaces for some metrizable uniformities 105

G. Paria: The non-symmetric punch problem in layered media by the Wiener-Hopf method 116

M. G. Barratt: Spaces of finite characteristic . . . 124

F. V. Atkinson, G. A. Watterson and **P. A. P. Moran**: A matrix inequality 137

E. J. Brody: On infinitely generated modules . . . 141

B. Kuttner: Some theorems on Mercer's and other related transformations 151

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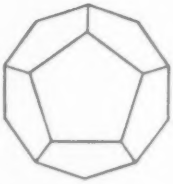

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SEMI-REGULAR RINGS AND SEMI-REGULAR IDEALS

By D. G. NORTHCOTT (*Sheffield*)

[Received 10 December 1958]

Introduction

A WELL-KNOWN theorem by F. S. Macaulay [(**3**) § 48] concerning ideals in a polynomial ring $K[X_1, X_2, ..., X_n]$, where K is a commutative field, asserts the following: *If an ideal* $\mathfrak{a}$ *of rank* r *can be generated by* r *elements, then the prime ideals belonging to* $\mathfrak{a}$ *all have rank equal to* r. Later the corresponding result for regular local rings was proved by I. S. Cohen [(**1**) 99 Theorem 21].

Recently our knowledge concerning Noetherian rings in which the Macaulay–Cohen theorem holds has been greatly extended. In the first instance, these extensions were accomplished with the aid of homological methods [see D. Rees (**8**)] and by techniques using multiplicity theory [see M. Nagata (**4**) and D. G. Northcott (**6**)]. Subsequently, however, it was shown by D. Rees and the author (**7**) that the main part of these results can be established in a comparatively elementary manner. The contents of this latter paper will therefore be used to form the foundations of the present one.

Now Macaulay's arguments yield more than the unmixed character of certain ideals, and his way of expressing this was to say that, for homogeneous ideals, the conclusions could be strengthened by saying that the ideals were *perfect*. At the time when Cohen obtained his results, no similar concept was available for abstract local rings.

The purpose of the present paper is to show how certain generalizations of Macaulay's theorems on unmixed ideals can be extended in a similar way. Instead of merely proving that, under appropriate conditions, an ideal is unmixed, I shall also show that it is *semi-regular*.† To avoid misunderstandings, it needs to be emphasized that in a polynomial ring $K[X_1, X_2, ..., X_n]$, with K a field, semi-regular ideals are *not* the same as perfect ideals. However, suppose that $\mathfrak{a}$ is a *homogeneous* ideal and Q is the ring of fractions formed with respect to the prime ideal $(X_1, X_2, ..., X_n)$. *Then* $\mathfrak{a}$ *is perfect in Macaulay's sense if and only if* $Q\mathfrak{a}$ *is a semi-regular* Q*-ideal* [cf. W. Gröbner (**2**) 199]. Again, in a

† The definition is given in § 1.

Quart. J. Math. Oxford (2), 11 (1960), 81–104.

regular local ring an ideal is semi-regular if and only if it is perfect in the sense of Rees [(**8**) Theorem 5.4]. I shall say no more on the connexions between the two concepts. There are strong indications suggesting that semi-regularity is the natural one to use in the type of theory developed here.

As remarked above, the basic ideas on which this paper rests have emerged only recently and, in fact, there has been considerable diversity in the terminology used, as well as in the methods employed in the various demonstrations. This makes it a somewhat tiresome matter to relate results found in different papers. It has therefore seemed worth while to include brief discussions of certain proofs which are already in the literature. In this way, apart from results set out in the short and elementary paper (**7**), the present account becomes self-contained.

1. Some definitions

To avoid constant repetition, it is to be understood that, when we speak of a *ring*, we shall always mean *a commutative ring with an identity element*. If R is a ring, then a *proper ideal* of R is any ideal different from R itself. R is said to be *Noetherian* if all its ideals are finitely generated.

Let R be a Noetherian ring. If a proper ideal $\mathfrak{a}$ can be generated by r ($r \geqslant 0$) elements† then, by a well-known theorem by W. Krull [see, for example, (**5**) 60, Theorem 7)], the rank of $\mathfrak{a}$ is at most r. The first formal definition extends the scope of a term used by Macaulay in connexion with polynomial rings.

DEFINITION. *A proper ideal* $\mathfrak{a}$ *which can be generated by* s *elements, where* $s = \operatorname{rank} \mathfrak{a}$, *will be said to be of the 'principal class'.*

As remarked in the introduction, Macaulay and Cohen were concerned with particular Noetherian rings in which ideals of the principal class are always unmixed with respect to rank. This prompts the next definition which, however, will be modified later for the sake of uniformity.

DEFINITION. *A Noetherian ring with the property that, for each ideal* $\mathfrak{a}$ *of the principal class, all the prime ideals belonging to* $\mathfrak{a}$ *have the same rank as* $\mathfrak{a}$ *will be called a 'Macaulay–Cohen ring'. (Abbreviation: MC-ring.)*

Let R be any ring and $\mathfrak{a}$ one of its proper ideals.

† The empty set is regarded as generating the zero ideal.

DEFINITION. *The ideal* $\mathfrak{a}$ *will be said to be 'semi-regular' if* $R/\mathfrak{a}$ *is an MC-ring.*

According to this definition, a ring R is an MC-ring if and only if its zero ideal is semi-regular. Now for local rings (i.e. Noetherian rings having only one maximal ideal) there is a notion of semi-regularity based on different considerations [see (7) 371]. To be explicit, a local ring Q of dimension d ($d \geqslant 0$) is said to be 'semi-regular' if there exist d non-units $a_1, a_2, \ldots, a_d$ such that

$$(a_1, a_2, \ldots, a_{i-1}) : a_i = (a_1, a_2, \ldots, a_{i-1})$$

for $i = 1, 2, \ldots, d$. (Note that every zero-dimensional local ring is semi-regular.) In other words, in the terminology of (7), Q is semi-regular if there exists a *prime sequence*† with d terms. However, it is a fundamental result [(7) 372 Theorem 2.4] that a local ring is an MC-ring if and only if it is semi-regular in the sense just explained. In view of this it will not lead to confusion if we rename MC-rings as follows:

DEFINITION. *An MC-ring, that is a Noetherian ring in which the Macaulay–Cohen theorem on unmixed ideals holds, will also be known as a 'semi-regular' ring.*

This secures the desideratum that a ring is semi-regular when and only when its zero ideal is semi-regular.

2. Preliminary observations

It is often convenient, when considering semi-regular rings and ideals, to transform a problem to a ring of fractions formed with respect to a suitable multiplicatively closed set. We make some observations concerning the principles governing this procedure.

LEMMA 1. *Let* R *be a commutative ring with an identity element and* S *a multiplicatively closed set of elements of* R *not containing zero. Further, let* $\mathfrak{a}$ *be a proper ideal not meeting* S. *Then, with the usual notation for rings of fractions, there exists a ring-isomorphism*

$$R_S/\mathfrak{a}_S \approx (R/\mathfrak{a})_{(S/\mathfrak{a})}.$$

Remarks. As is customary, $\mathfrak{a}_S$ denotes the ideal $R_S\mathfrak{a}$ and, as is clear, this ideal is proper. By $S/\mathfrak{a}$ is to be understood the natural image of S in $R/\mathfrak{a}$. When S is the complement of a proper prime ideal $\mathfrak{p}$

† The notion of a prime sequence is available for any ring. Quite precisely, $a_1, a_2, \ldots, a_d$ is a prime sequence if $(a_1, a_2, \ldots, a_d)$ is a proper ideal and

$$(a_1, \ldots, a_{i-1}) : a_i = (a_1, \ldots, a_{i-1})$$

for each i.

containing $\mathfrak{a}$, we shall sometimes write $R_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p}$ in place of R_S and $\mathfrak{a}_S$. The above isomorphism then becomes

$$R_\mathfrak{p}/\mathfrak{a}_\mathfrak{p} \approx (R/\mathfrak{a})_{(\mathfrak{p}/\mathfrak{a})}.$$

Proof. A proof is given in [(**9**) 226]. A shorter version is as follows. Write the elements of R_S as formal quotients $[r/c]$, where $r \in R$, $c \in S$, and we have the usual rule for identification. A homomorphism of R_S on to $(R/\mathfrak{a})_{(S/\mathfrak{a})}$ is obtained by mapping $[r/c]$ into $[\phi(r)/\phi(c)]$, where $\phi\colon R \to R/\mathfrak{a}$ is the natural homomorphism. A trivial verification then shows that the homomorphism of R_S has kernel $\mathfrak{a}_S$.

Let R be a Noetherian ring. It is known [see (**7**) 373 Theorem 3.1 and (**4**) 213 Proposition 8] that R is semi-regular if and only if $R_\mathfrak{m}$ is semi-regular for every maximal ideal $\mathfrak{m}$. Again, *suppose that* $\mathfrak{p}$ *is a proper prime ideal and that* $\mathfrak{p}$ *contains a prime sequence with* r *terms. Then* $R_\mathfrak{p}$ *is semi-regular*, for it is a local ring whose maximal ideal contains a prime sequence having dim $R_\mathfrak{p}$ terms.

Next assume that R is semi-regular and that S is a multiplicatively closed set not containing zero. The ring of fractions of R_S with respect to a maximal ideal can be identified with $R_\mathfrak{p}$, where $\mathfrak{p}$ is the contraction of the maximal ideal in R [see, for example, (**9**) 231]. However, $R_\mathfrak{p}$ is semi-regular [(**7**) 373 Theorem 3.1] and this shows that R_S is semi-regular.

The first theorem and its corollary combine some of these observations with Lemma 1.

THEOREM 1. *Let R be a commutative ring with an identity element and $\mathfrak{a}$ one of its proper ideals. If now $R/\mathfrak{a}$ is Noetherian, then $\mathfrak{a}$ is semi-regular if and only if $\mathfrak{a}_\mathfrak{m}$ is a semi-regular $R_\mathfrak{m}$-ideal for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$.*

COROLLARY. *If $\mathfrak{a}$ is a semi-regular R-ideal and S is a multiplicatively closed set not meeting $\mathfrak{a}$, then $\mathfrak{a}_S$ is a semi-regular R_S-ideal.*

Consider a semi-regular ideal $\mathfrak{a}$ in a ring R. Since $R/\mathfrak{a}$ is Noetherian, $\mathfrak{a}$ possesses a primary decomposition, and therefore we may speak of the 'prime ideals' belonging to $\mathfrak{a}$. Also the ring $R/\mathfrak{a}$ is semi-regular and therefore its zero ideal, which is certainly of the principal class, has no embedded prime ideals. These remarks yield our next result:

THEOREM 2. *A semi-regular ideal in a ring R has no embedded prime ideals.*

THEOREM 3. *If A and B are semi-regular rings, then their direct sum $R = A + B$ is also semi-regular.*

Proof. R is Noetherian since it is the direct sum of two Noetherian rings. Now let M be a maximal ideal of R; then either $M = \mathfrak{m}+B$, where $\mathfrak{m}$ is a maximal A-ideal, or $M = A+\mathfrak{n}$, where $\mathfrak{n}$ is a maximal B-ideal. Suppose, for definiteness, that the former is true; then R_M is isomorphic to $A_\mathfrak{m}$ and so is semi-regular. The theorem follows.

As an example, let A be a one-dimensional regular local ring with maximal ideal $\mathfrak{m}$ and let B be a field. Then $R = A+B$ is semi-regular. Further, $M = \mathfrak{m}+B$ and $M' = A+0$ are the only maximal ideals of R and we have $R_M \approx A$, $R_{M'} \approx B$. Now every proper ideal of A or of B is semi-regular; hence, by Theorem 1, every proper R-ideal is semi-regular. In particular, $M \cap M'$ is semi-regular, and this shows that, in a semi-regular ring, the prime ideals belonging to a semi-regular ideal need not all have the same rank. However, the situation is less complex than this observation might suggest.

THEOREM 4. *Let R be a semi-regular ring and $\mathfrak{a}$ a semi-regular R-ideal. If now $\mathfrak{p}$ and $\mathfrak{p}'$ are prime ideals belonging to $\mathfrak{a}$ which are both contained in the same maximal ideal $\mathfrak{m}$, then* $\operatorname{rank}\mathfrak{p} = \operatorname{rank}\mathfrak{p}'$.

Proof. The ideal $\mathfrak{a}_\mathfrak{m}$ is semi-regular (Theorem 1). Hence $R_\mathfrak{m}/\mathfrak{a}_\mathfrak{m}$ is a semi-regular local ring and $\mathfrak{p}_\mathfrak{m}/\mathfrak{a}_\mathfrak{m}$, $\mathfrak{p}'_\mathfrak{m}/\mathfrak{a}_\mathfrak{m}$, being prime ideals belonging to its zero ideal, both have zero rank. It follows [(7) 372 Theorem 2.3] that $\mathfrak{p}_\mathfrak{m}/\mathfrak{a}_\mathfrak{m}$ and $\mathfrak{p}'_\mathfrak{m}/\mathfrak{a}_\mathfrak{m}$ have the same dimension, and hence that $\dim\mathfrak{p}_\mathfrak{m} = \dim\mathfrak{p}'_\mathfrak{m}$. But $R_\mathfrak{m}$ is also a semi-regular local ring. Consequently, by the same result, $\operatorname{rank}\mathfrak{p}_\mathfrak{m} = \operatorname{rank}\mathfrak{p}'_\mathfrak{m}$ and now the theorem follows.

THEOREM 5. *Let $\mathfrak{a}$ be a semi-regular ideal in a ring R and $\mathfrak{p}$, $\mathfrak{p}'$ proper prime ideals such that $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{p}'$. Then*

$$\operatorname{rank}(\mathfrak{p}'/\mathfrak{a}) = \operatorname{rank}(\mathfrak{p}'/\mathfrak{p})+\operatorname{rank}(\mathfrak{p}/\mathfrak{a}).$$

Proof. By going to the ring $R_{\mathfrak{p}'}/\mathfrak{a}_{\mathfrak{p}'}$ we reduce the theorem to the case in which R is a semi-regular local ring, with maximal ideal $\mathfrak{p}'$, and $\mathfrak{a}$ is the zero ideal. For this situation the theorem is known [(7) 372 Theorem 2.3].

COROLLARY 1. *With the same assumptions as in the theorem, let*

$$\mathfrak{a} \subseteq \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_k = \mathfrak{p}'$$

be any chain of prime ideals such that (i) *$\mathfrak{p}_0$ is a minimal prime ideal of $\mathfrak{a}$ and* (ii) *for each value of i ($i = 0, 1, \ldots, k-1$) there is no prime ideal strictly between $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$. Then $k = \operatorname{rank}(\mathfrak{p}'/\mathfrak{a})$.*

Since $R/\mathfrak{a}$ is Noetherian, we can speak of the minimal prime ideals of $\mathfrak{a}$. The corollary follows [see (7) 373 Theorem 4.1] again by passage to the ring $R_{\mathfrak{p}'}/\mathfrak{a}_{\mathfrak{p}'}$.

Still keeping to the same notation and hypotheses, we obtain a further result by combining the corollary with the theorem.

COROLLARY 2. *For any chain*

$$\mathfrak{p} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_m = \mathfrak{p}'$$

of prime ideals such that no further prime ideal can be inserted between two consecutive terms, we have $m = \operatorname{rank}(\mathfrak{p}'/\mathfrak{p})$.

The final theorem of this section embodies a remark of Nagata's [(4) 211 Remark 1]. This particular result will be generalized both in § 4 and in § 6.

THEOREM 6. *Let R be a semi-regular ring and $\mathfrak{a}$ an ideal of the principal class. Then $R/\mathfrak{a}$ is semi-regular.*

Proof. Let $\mathfrak{a} = (a_1, a_2, \ldots, a_r)$ be of rank r and $c_1, c_2, \ldots, c_s$ such that, if $\mathfrak{b} = (a_1, \ldots, a_r, c_1, \ldots, c_s)$, then $\mathfrak{b}/\mathfrak{a}$ is a proper ideal of rank s. If now $\mathfrak{p}'$ is a prime ideal belonging to $\mathfrak{b}$, then we wish to show that

$$\operatorname{rank}(\mathfrak{p}'/\mathfrak{a}) = s.$$

Choose a prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$ and

$$\operatorname{rank}(\mathfrak{p}'/\mathfrak{a}) = \operatorname{rank}(\mathfrak{p}'/\mathfrak{p}).$$

Then, applying Theorem 5 to $\mathfrak{p}$, $\mathfrak{p}'$ and the zero ideal of R, we get

$$\operatorname{rank}(\mathfrak{p}'/\mathfrak{a}) = \operatorname{rank}\mathfrak{p}' - \operatorname{rank}\mathfrak{p} = \operatorname{rank}\mathfrak{p}' - r. \qquad \text{(A)}$$

In particular, $\operatorname{rank}\mathfrak{p}' = r + \operatorname{rank}(\mathfrak{p}'/\mathfrak{a}) \geqslant r+s$.

This last observation makes it clear that $\mathfrak{b}$ has rank $r+s$ and is therefore of the principal class. Accordingly rank $\mathfrak{p}' = r+s$ and now the required result follows from (A).

3. Adjunction of indeterminates

Techniques involving the adjunction of indeterminates to a ring are comparatively familiar, but the use I propose to make of this idea has one or two slightly unusual features.

Throughout this section, R will denote a commutative ring with an identity element, and $x_1, x_2, \ldots, x_p$ are indeterminates. Consider elements $\phi(x_1, x_2, \ldots, x_p)$ of $P = R[x_1, x_2, \ldots, x_p]$ with the property that *the R-ideal generated by the coefficients of ϕ is R itself.* It is easy to see that these elements form a *multiplicatively closed* set S. Further, we contend that *no element of S is a zero-divisor in P.*

To establish the second assertion,† suppose that

$$\phi(x_1, x_2, \ldots, x_p)g(x_1, x_2, \ldots, x_p) = 0,$$

where $\phi \in S$ and at least one coefficient of g is not zero. Let R_0 be the subring of R generated by the identity element and the coefficients of ϕ and g. Then R_0 is Noetherian and ϕ is a zero-divisor in $R_0[x_1, x_2, \ldots, x_p]$. Hence ϕ belongs to the extension, to $R_0[x_1, x_2, \ldots, x_p]$, of some prime ideal belonging to the zero ideal of R_0. It follows that there exists $c \neq 0$ $(c \in R_0)$ such that $c\phi(x_1, \ldots, x_p) = 0$. But, since the coefficients of ϕ generate R, this implies that $c = 0$ and so we have a contradiction.

Now let us put

$$R' = P_S = \{R[x_1, x_2, \ldots, x_p]\}_S.$$

It is easy to check that, if $\mathfrak{a}$ and $\mathfrak{b}$ are R-ideals, then

$$R'\mathfrak{a} \cap R = \mathfrak{a}, \qquad R'(\mathfrak{a} \cap \mathfrak{b}) = R'\mathfrak{a} \cap R'\mathfrak{b}.$$

Again, if $\qquad f(x_1, x_2, \ldots, x_p) \in P, \quad \psi(x_1, x_2, \ldots, x_p) \in S,$

and all the coefficients of $f\psi$ belong to $\mathfrak{a}$, then already the coefficients of f must be in $\mathfrak{a}$. (This is because, if we use a bar to denote the effect of applying the natural mapping $R \to R/\mathfrak{a}$, then

$$\bar{f}(x_1, \ldots, x_p)\bar{\psi}(x_1, x_2, \ldots, x_p) = 0$$

and, since the coefficients of $\bar{\psi}$ generate $R/\mathfrak{a}$, $\bar{\psi}(x_1, \ldots, x_p)$ is not a zero-divisor.) It now follows without difficulty that

$$R'(\mathfrak{a} : \mathfrak{b}) = R'\mathfrak{a} : R'\mathfrak{b}.$$

On the basis of these results, we shall be able to transfer to R' certain problems concerning the ideals of R. The advantage gained is that the indeterminates may be used to form 'general elements' of the ideals under consideration.

Since it is sometimes convenient to adjoin sets of variables in succession, rather than all at once, we make a brief analysis of this situation. Consider a set

$$f_1(x_1, \ldots, x_p), \quad f_2(x_1, \ldots, x_p), \quad \ldots, \quad f_t(x_1, \ldots, x_p)$$

of elements of $R[x_1, x_2, \ldots, x_p]$. If the coefficients of these polynomials generate the R-ideal $\mathfrak{a}$, then

$$R'f_1 + \ldots + R'f_t \subseteq R'\mathfrak{a}.$$

Since $R'\mathfrak{a} \cap R = \mathfrak{a}$, we see that, if $\mathfrak{a}$ is proper, then so is $R'f_1 + \ldots + R'f_t$.

† The argument, which follows, is based on Cohen [(**1**) 85 Lemma 14].

On the other hand, if $\mathfrak{a} = R$, then by choosing suitable integers m_1, m_2,..., m_t we can make

$$x_1^{m_1} f_1 + x_1^{m_2} f_2 + \ldots + x_1^{m_t} f_t$$

an element of S, and this shows that

$$R'f_1 + \ldots + R'f_t = R'.$$

Thus, to sum up,

$$f_1(x_1,\ldots,x_p), \quad f_2(x_1,\ldots,x_p), \quad \ldots, \quad f_t(x_1,\ldots,x_p)$$

generate a proper R'-ideal if and only if their coefficients generate a proper R-ideal.

To bring out the connexion of R' with R and the indeterminates x_1, x_2,..., x_p, let us write $R' = R_{(x)}$.

Suppose now that y_1, y_2,..., y_q are further indeterminates. Then, after the concluding remark of the last paragraph, it is easy to see that

$$R_{(x,y)} = (R_{(x)})_{(y)},$$

where (x, y) denotes the complete set $x_1,\ldots, x_p, y_1,\ldots, y_q$.

Next suppose that R is Noetherian. Let $\mathfrak{a} = (a_1, a_2,\ldots, a_m)$ be a proper R-ideal of rank k, and x_{ij} $(1 \leqslant i, j \leqslant m)$ indeterminates. Put

$$a'_j = \sum_{i=1}^{m} a_i x_{ij} \quad (1 \leqslant j \leqslant m)$$

and $R' = R_{(\ldots x_{ij}\ldots)}$. Then R' is Noetherian and

$$R'(a'_1,\ldots, a'_m) = R'(a_1,\ldots, a_m)$$

because, in R', the determinant $|x_{ij}|$ is a unit. Also, since S will not meet any prime ideal belonging to the ideal generated by a_1, a_2,..., a_m in $R[\ldots x_{ij}\ldots]$, $R'(a_1, a_2,\ldots, a_m)$ and $R(a_1, a_2,\ldots, a_m)$ have similar primary decompositions. In particular they have the same rank, namely k. We now contend that, *if $1 \leqslant n \leqslant k$, then any n elements among a'_1, a'_2,..., a'_m generate an R'-ideal of rank n.* This is easily seen to be true when $n = 1$. We therefore suppose that $1 < n \leqslant k$ and that we already know that each set of $n-1$ elements among a'_1, a'_2,..., a'_m generate an ideal of rank $n-1$.

Let s_1, s_2,..., s_n be any n distinct integers from among 1, 2,..., m and assume that $(a'_{s_1}, a'_{s_2},\ldots, a'_{s_n})$ has rank *less* than n. Then there exists a prime R'-ideal, $\mathfrak{p}'$ say, of rank $n-1$ and containing $(a'_{s_1}, a'_{s_2},\ldots, a'_{s_n})$. Now $\mathfrak{p}'$ will *belong* to $(a'_{s_1}, a'_{s_2},\ldots, a'_{s_{n-1}})$, which shows that $\mathfrak{p}'$ can be generated by elements ϕ_1, ϕ_2,..., ϕ_n where each ϕ_i is a polynomial (with coefficients in R) involving *only* the $(n-1)m$ variables that occur in the expressions for $a'_{s_1}, a'_{s_2},\ldots, a'_{s_{n-1}}$. Let s be any integer in the range 1, 2,..., m

different from all of $s_1, s_2, \ldots, s_n$. By suitably permuting the variables we can obtain an isomorphism of R' on to itself which (i) leaves every element of R fixed; (ii) interchanges the variables concerned in the representations of a'_{s_n} and a'_s so that the former is mapped into the latter; (iii) leaves all the other variables unchanged. This isomorphism maps $\mathfrak{p}'$ on to itself and $(a'_{s_1}, \ldots, a'_{s_{n-1}}, a'_{s_n})$ on to $(a'_{s_1}, \ldots, a'_{s_{n-1}}, a'_s)$. Consequently

$$(a'_{s_1}, \ldots, a'_{s_{n-1}}, a'_s) \subseteq \mathfrak{p}'.$$

Repeating this interchange argument, we find that *any* n elements among $a'_1, a'_2, \ldots, a'_m$ generate an ideal contained in $\mathfrak{p}'$. Accordingly

$$R'(a'_1, a'_2, \ldots, a'_m) \subseteq \mathfrak{p}',$$

which is impossible since the rank of $R'(a'_1, a'_2, \ldots, a'_m)$ is k. This is the required contradiction.

The usefulness of the above considerations in the special theory we are discussing, stems from the following result.

PROPOSITION 1. *Let R be a semi-regular ring and $x_1, x_2, \ldots, x_p$ indeterminates. Then, with the above notation, $R_{(x)}$ is also semi-regular.*

Since the property of being semi-regular is preserved in the transition to a ring of fractions (Theorem 1), Proposition 1 is an immediate consequence of the definitions and the following beautiful theorem by Nagata [(4) 213 Proposition 9]: *if R is semi-regular, then the polynomial ring $R[x_1, x_2, \ldots, x_p]$ is also semi-regular.* For the sake of completeness, I sketch Nagata's proof. In the first place it is necessary only to consider $R[z]$, where z is a single indeterminate. Let $\mathfrak{M}$ be a maximal ideal of $R[z]$; then it suffices to show that the ring of fractions of $R[z]$ with respect to $\mathfrak{M}$ is semi-regular. By first forming fractions with respect to $R-(R \cap \mathfrak{M})$, this problem is reduced to the case in which R is a semi-regular local ring and $\mathfrak{M}$ contracts to its maximal ideal $\mathfrak{m}$ say. Let the dimension of the local ring R be d; then there exist non-units $a_1, a_2, \ldots, a_d$ which form a prime sequence in R and therefore also in $R[z]$. Now, in $R[z]$, $(a_1, a_2, \ldots, a_d)$ is a primary ideal belonging to $\mathfrak{m}R[z]$, and we can find β in $\mathfrak{M}$ which is not in $\mathfrak{m}R[z]$. Then $a_1, \ldots, a_d, \beta$ is a prime sequence and, since there is clearly no prime ideal between $\mathfrak{m}R[z]$ and $\mathfrak{M}$, $\mathfrak{M}$ is a minimal prime ideal belonging to $(a_1, a_2, \ldots, a_d, \beta)$. This shows that $\mathfrak{M}$ has rank $d+1$ and contains a prime sequence with $d+1$ terms. The required result therefore follows.

4. Ideals of the principal class

After the general discussion of the last two sections we can build up, for semi-regular rings, a theory similar to that developed by Macaulay

for polynomial rings. Indeed, from now on, several of the proofs will be suitably modified versions of the original demonstrations.

THEOREM 7. *Let R be a semi-regular ring and $\mathfrak{a}$ an ideal of the principal class. Then every power of $\mathfrak{a}$ is semi-regular.*

It is convenient to begin by establishing a weaker result namely [cf. (3) 51]

LEMMA 2. *With assumptions as in Theorem 7, each power $\mathfrak{a}^s$ $(s \geqslant 1)$ of $\mathfrak{a}$ is free from embedded prime ideals and therefore all the prime ideals belonging to $\mathfrak{a}^s$ have the same rank.*

Proof. Let $\mathfrak{a} = (a_1, a_2, \ldots, a_m)$, where $m = \operatorname{rank} \mathfrak{a}$. The proof will be by induction with respect to m, and we can observe at once that the lemma is obvious for $m = 0$ and $m = 1$. From now on, therefore, we shall suppose that $m \geqslant 2$ and that the lemma has been proved, in all cases, for ideals of the principal class whose rank is less than m.

Turning now to the consideration of $\mathfrak{a} = (a_1, a_2, \ldots, a_m)$, by forming m linear combinations of $a_1, a_2, \ldots, a_m$ with indeterminate coefficients (see § 3), we can suppose (for the purposes of the proof) that any p $(1 \leqslant p \leqslant m)$ elements among $a_1, a_2, \ldots, a_m$ generate an ideal of rank p. This will be a great advantage.

It remains to be shown that $\mathfrak{a}^s$ $(s \geqslant 1)$ has no embedded prime ideals, and for this we use induction with respect to s. When $s = 1$, there is no problem. We shall therefore assume that $s > 1$ and that it has already been shown that $\mathfrak{a}^{s-1}$ has no embedded prime ideals. It will now be supposed that $\mathfrak{a}^s$ has an embedded prime ideal $\mathfrak{p}$, and from this we shall obtain a contradiction. This will complete the proof.

By the inductive hypothesis, all the prime ideals of $\mathfrak{a}^{s-1}$ are minimal and therefore they are the same as the minimal prime ideals belonging to $\mathfrak{a}^s$. Accordingly, $\mathfrak{p}$ is not contained in any prime ideal of $\mathfrak{a}^{s-1}$, and so we can find $u \in \mathfrak{p}$ such that $\mathfrak{a}^{s-1} : u = \mathfrak{a}^{s-1}$ while, of course, $\mathfrak{a}^s : u$ strictly contains $\mathfrak{a}^s$.

Choose v not in $\mathfrak{a}^s$ with the property that $uv \in \mathfrak{a}^s$. Since $\mathfrak{a}^s \subseteq \mathfrak{a}^{s-1}$, we have $v \in \mathfrak{a}^{s-1} : u = \mathfrak{a}^{s-1}$. Hence v can be written in the form

$$v = a_1 \phi_1(a_1, \ldots, a_m) + a_2 \phi_2(a_2, \ldots, a_m) + \ldots + a_m \phi_m(a_m),$$

where $\phi_i(a_i, a_{i+1}, \ldots, a_m)$ is a form of degree $s-2$ in $a_i, a_{i+1}, \ldots, a_m$ alone, its coefficients being in R. But $uv \in \mathfrak{a}^s$; consequently

$$uv \equiv a_1 z \pmod{(a_2, a_3, \ldots, a_m)^s},$$

where $z \in \mathfrak{a}^{s-1}$, and therefore

$$ua_1 \phi_1 \equiv a_1 z \pmod{(a_2, a_3, \ldots, a_m)^{s-1}}.$$

But the first inductive hypothesis shows that all the prime ideals of $(a_2, a_3, ..., a_m)^{s-1}$ have rank $m-1$. Hence a_1 does not belong to any of them, and therefore

$$u\phi_1 \equiv z \pmod{(a_2, a_3, ..., a_m)^{s-1}}.$$

Accordingly $u\phi_1 \in \mathfrak{a}^{s-1}$, and now it follows that $\phi_1 \in \mathfrak{a}^{s-1}$.

Put $$v' = a_2\phi_2(a_2, ..., a_m) + a_3\phi_3(a_3, ..., a_m) + ... + a_m\phi(a_m).$$

Then $v' \notin \mathfrak{a}^s$ but $uv' \in \mathfrak{a}^s$, so that v' has properties and a representation similar to those of v. A repetition of the argument just given then shows that $\phi_2(a_2, ..., a_m)$ also belongs to $\mathfrak{a}^{s-1}$. Proceeding in this way we eventually find that each $\phi_i(a_i, a_{i+1}, ..., a_m)$ belongs to $\mathfrak{a}^{s-1}$, and therefore $v \in \mathfrak{a}^s$. This is the required contradiction.

Proof of Theorem 7. By virtue of Theorem 1, it will suffice to prove the theorem for semi-regular local rings. Accordingly, from now on, this extra condition will be imposed on R. Denote its maximal ideal by $\mathfrak{m}$ and put $d = \dim R$. As before we suppose that $\mathfrak{a}$ has rank m.

We shall construct, in succession, non-units $b_1, b_2, ..., b_{d-m}$ such that

$$(\mathfrak{a}^s, b_1, ..., b_{i-1}) : b_i = (\mathfrak{a}^s, b_1, ..., b_{i-1}) \quad (i = 1, 2, ..., d-m).$$

Since the dimension of $\mathfrak{a}^s$ is $d-m$ [(**7**) 372 Theorem 2.3], this will prove that $R/\mathfrak{a}^s$ is semi-regular and thereby establish the theorem.

Assume for the moment that $d > m$. By Lemma 2, $\mathfrak{a}^s$ has no embedded prime ideals, and hence $\mathfrak{m}$ is not a prime ideal of $\mathfrak{a}^s$. We can therefore find b_1 in $\mathfrak{m}$ so that $(0) : b_1 = (0)$ and $\mathfrak{a}^s : b_1 = \mathfrak{a}^s$. Now $R/(b_1)$ is a $(d-1)$-dimensional semi-regular local ring (Theorem 6) and $(\mathfrak{a}, b_1)$ has dimension $d-m-1$. Consequently $(\mathfrak{a}, b_1)/(b_1)$ has rank m and therefore it belongs to the principal class. It follows, by Lemma 2, that $(\mathfrak{a}^s, b_1)/(b_1)$ has no embedded prime ideals, and this implies that $(\mathfrak{a}^s, b_1)$ is also free from embedded prime ideals. If therefore $d > m+1$, $\mathfrak{m}$ will not belong to $(\mathfrak{a}^s, b_1)$, which is of rank $m+1$, and we shall be able to find b_2 in $\mathfrak{m}$ such that $(\mathfrak{a}^s, b_1) : b_2 = (\mathfrak{a}^s, b_1)$ and $(b_1) : b_2 = (b_1)$. This time $(\mathfrak{a}, b_1, b_2)/(b_1, b_2)$ is an ideal of the principal class in the $(d-2)$-dimensional semi-regular local ring $R/(b_1, b_2)$; hence, again by Lemma 2, $(\mathfrak{a}^s, b_1, b_2)$ has no embedded prime ideals. Proceeding in this way we are able to construct $b_1, b_2, ..., b_{d-m}$ with the required properties.

THEOREM 8. *Let R be a semi-regular ring and $a_1, a_2, ..., a_m$ $(m \geqslant 1)$ elements which generate an ideal of the principal class whose rank is m. If now $L_1, L_2, ..., L_m$ are elements of R such that*

$$a_1 L_1 + a_2 L_2 + ... + a_m L_m = 0, \tag{B}$$

then they can be expressed in the form

$$L_i = \sum_{j=1}^{m} D_{ij} a_j \quad (1 \leqslant i \leqslant m), \tag{C}$$

where $D_{ij} \in R$ *and*

$$D_{ii} = 0, \qquad D_{ij} = -D_{ji} \quad (i \neq j). \tag{D}$$

Conversely, elements L_i *defined by* (C), *where the* D_{ij} *satisfy* (D), *have the property* (B).

Proof. The converse is trivial, and I shall say no more about it. The remaining part of the theorem will be proved by induction with respect to m. Since the assertion is obvious for $m = 1$, we shall assume that $m \geqslant 2$ and that the theorem has been proved for all smaller values of the inductive variable.

Let x_{ij} $(1 \leqslant i, j \leqslant m)$ be indeterminates and put

$$\alpha_j = \sum_{i=1}^{m} a_i x_{ij} \quad (1 \leqslant j \leqslant m).$$

Then

$$\sum_{j=1}^{m} \alpha_j X_{kj} = |X| a_k,$$

where $|X|$ is the determinant $|x_{ij}|$ and X_{kj} is the cofactor of x_{kj}. Consequently, if we put

$$\Lambda_j = \sum_{k=1}^{m} L_k X_{kj} \quad (1 \leqslant j \leqslant m),$$

then $\alpha_1 \Lambda_1 + \alpha_2 \Lambda_2 + \ldots + \alpha_m \Lambda_m = 0$.

We pass now to the semi-regular ring $P = R_{(x)}$ [see Proposition 1], where (x) denotes the full set of indeterminates x_{ij}. Observe that, in this ring, any t $(t \leqslant m)$ elements among $\alpha_1, \alpha_2, \ldots, \alpha_m$ generate an ideal of rank t [see § 3].

Put $P^* = P/P\alpha_m$; then P^* is semi-regular (Theorem 6). Further, using an asterisk to denote residues to modulus $P\alpha_m$, we see that

$$\alpha_1^* \Lambda_1^* + \alpha_2^* \Lambda_2^* + \ldots + \alpha_{m-1}^* \Lambda_{m-1}^* = 0$$

and $(\alpha_1^*, \alpha_2^*, \ldots, \alpha_{m-1}^*)$ is an ideal of rank $m-1$ belonging to the principal class.† Accordingly, by the inductive hypothesis, we can find elements Δ_{ij} $(1 \leqslant i, j \leqslant m-1)$ in $R_{(x)}$ such that

$$\Delta_{ii} = 0 \quad (\text{all } i), \qquad \Delta_{ij} = -\Delta_{ji} \quad (i \neq j),$$

and

$$\Lambda_i^* = \sum_{j=1}^{m-1} \Delta_{ij}^* \alpha_j^* \quad (1 \leqslant i \leqslant m-1).$$

† This follows from Theorem 5.

Now choose elements $\Delta_{1m}, \Delta_{2m}, \ldots, \Delta_{(m-1)m}$ in $R_{(x)}$ such that

$$\Lambda_i = \sum_{j=1}^{m-1} \Delta_{ij}\alpha_j + \Delta_{im}\alpha_m \quad (1 \leqslant i \leqslant m-1).$$

Then

$$\begin{aligned} 0 &= \Lambda_1\alpha_1 + \ldots + \Lambda_{m-1}\alpha_{m-1} + \Lambda_m\alpha_m \\ &= \Delta_{1m}\alpha_1\alpha_m + \ldots + \Delta_{(m-1)m}\alpha_{m-1}\alpha_m + \Lambda_m\alpha_m. \end{aligned}$$

But α_m is not a zero-divisor in P, and so

$$\Lambda_m = (-\Delta_{1m})\alpha_1 + (-\Delta_{2m})\alpha_2 + \ldots + (-\Delta_{(m-1)m})\alpha_{m-1}.$$

Thus, if we put $\Delta_{mi} = -\Delta_{im}$ $(i < m)$ and $\Delta_{mm} = 0$, then

$$\Lambda_i = \sum_{j=1}^{m} \Delta_{ij}\alpha_j \quad (1 \leqslant i \leqslant m), \qquad \Delta_{ii} = 0,$$

$$\Delta_{ij} = -\Delta_{ji} \quad (\text{for } i \neq j).$$

Next $\sum_{i=1}^{m} \Lambda_i x_{\mu i} = |X| L_\mu$, and therefore

$$|X| L_\mu = \sum_{i,j} \Delta_{ij} x_{\mu i}\alpha_j = \sum_{i,j,\nu} \Delta_{ij} x_{\mu i} x_{\nu j} a_\nu = \sum_{\nu=1}^{m} \tilde{D}_{\mu\nu} a_\nu,$$

where

$$\tilde{D}_{\mu\nu} = \sum_{i,j} \Delta_{ij} x_{\mu i} x_{\nu j}.$$

Clearly $\tilde{D}_{\mu\mu} = 0$ and $\tilde{D}_{\mu\nu} = -\tilde{D}_{\nu\mu}$. Now there exists a polynomial $\phi(\ldots x_{ij}\ldots)$ whose coefficients generate R, such that $\phi\tilde{D}_{\mu\nu} = \hat{D}_{\mu\nu}$ (say) belongs to $R[\ldots x_{ij}\ldots]$ for all μ, ν and we have $|X|\phi L_\mu = \sum_{\nu=1}^{m} \hat{D}_{\mu\nu} a_\nu$.

By equating coefficients we arrive at relations $c^{(l)} L_\mu = \sum_{\nu} \hat{D}^{(l)}_{\mu\nu} a_\nu$, where $c^{(l)}$ is a typical coefficient of $|X|\phi$ and $\hat{D}^{(l)}_{\mu\nu}$ is the corresponding coefficient of $\hat{D}_{\mu\nu}$. Clearly the $\hat{D}^{(l)}_{\mu\nu}$ enjoy the same antisymmetry properties as the $\tilde{D}_{\mu\nu}$. Finally, there exist elements $r^{(l)}$ in R such that $\sum_l r^{(l)} c^{(l)} = 1$ and then $L_\mu = \sum_\nu D_{\mu\nu} a_\nu$, where $D_{\mu\nu} = \sum_l r^{(l)} \hat{D}^{(l)}_{\mu\nu}$. The theorem follows.

5. Ideals defined by matrices (preliminaries)

Consider a matrix

$$\begin{Vmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1p} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{mp} \end{Vmatrix} \quad (m \leqslant p), \qquad \text{(E)}$$

in which the number of rows does not exceed the number of columns and the elements belong to a ring R. The m-rowed minors of (E) will

generate an ideal which in the sequel will be denoted by

$$\begin{pmatrix} a_{11} & a_{12} & . & . & a_{1p} \\ a_{21} & a_{22} & . & . & a_{2p} \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mp} \end{pmatrix}. \tag{F}$$

In other words, double vertical lines will be used to indicate a matrix while the same elements enclosed in round brackets will signify the ideal which the matrix determines. Note that, when $m = 1$, this notation agrees with the one whereby $(a_1, a_2, ..., a_p)$ is used for the ideal generated by the elements $a_1, a_2, ..., a_p$.

The ideal (F) remains unchanged if we permute either the rows or the columns amongst themselves. Also we may add to the elements of one row (column) a multiple, by an element of R, of some other row (column) without altering the ideal.

THEOREM 9. *Let R be a semi-regular ring and suppose that the ideal* (F) *is a proper R-ideal. Then its rank is at most $p-m+1$. Indeed each of its minimal prime ideals has rank not exceeding $p-m+1$.*

Remarks. The author has in his possession notes of a talk given by I. S. Cohen in which Cohen dealt with this theorem in the case where R is an arbitrary *local domain*. The same talk also contained, for *regular local rings*, a result which is connected with Theorem 11 of § 6. Unfortunately, although the notes contain brief indications of proofs which I have found very helpful, I was not able to reconstruct the original arguments. In all probability Cohen's treatment contained some short cuts which have been overlooked here.

Proof. Let us use the letter $\mathfrak{a}$ to denote the ideal determined by the matrix $\|a_{ij}\|$, where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant p$, and $m \leqslant p$. Further let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$. Then $R_{\mathfrak{p}}$ is a semi-regular ring and $\mathfrak{a}_{\mathfrak{p}}$, besides being defined by a matrix with m rows and p columns, is a primary ideal belonging to the maximal ideal. Thus, if we establish the first part of the theorem, we shall be in a position to conclude that $\dim R_{\mathfrak{p}} = \operatorname{rank} \mathfrak{a}_{\mathfrak{p}} \leqslant p-m+1$, and hence that $\operatorname{rank} \mathfrak{p} \leqslant p-m+1$. In other words, the second part of the theorem follows from the first and therefore I shall say no more about it.

The proof of the first part is by induction on m. For $m = 1$ the result is already known. We shall therefore suppose that $m \geqslant 2$ and that the first part of the theorem has been established for ideals defined by matrices with $m-1$ rows and at least $m-1$ columns. Further, in

accomplishing the inductive step we may suppose that R is a semi-regular local ring with maximal ideal $\mathfrak{m}$ (say).

Now, if any of the elements a_{ij} is a unit in R, then $\mathfrak{a}$ can be obtained from a matrix with $m-1$ rows and $p-1$ columns, and consequently, by the inductive hypothesis, its rank is at most $p-m+1$ as required. *Thus we may restrict our attention to the case in which all the a_{ij} belong to* $\mathfrak{m}$.

Consider the ideal

$$\mathfrak{b}' = \begin{pmatrix} a_{11}+t & a_{12} & . & . & a_{1p} \\ a_{21} & a_{22} & . & . & a_{2p} \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mp} \end{pmatrix}$$

in the ring $R[t]$, where t is an indeterminate. Since $m \geqslant 2$ and all the a_{ij} are in $\mathfrak{m}$, it follows that $\mathfrak{b}' \subseteq \mathfrak{m}R[t]$. The ring of fractions of $R[t]$ with respect to $\mathfrak{m}R[t]$ is semi-regular,† and in this ring $a_{11}+t$ is a unit. By an argument already used above, we can now conclude that the rank of the extension of $\mathfrak{b}'$ does not exceed $p-m+1$. This in turn implies that there is a prime ideal $\mathfrak{p}'$, belonging to $\mathfrak{b}'$ and contained in $\mathfrak{m}R[t]$, such that $\operatorname{rank}\mathfrak{p}' \leqslant p-m+1$.

In the semi-regular ring $R[t]$, the ideal $(\mathfrak{p}', t)$ is proper and, by a simple application of Theorem 5,

$$\operatorname{rank}(\mathfrak{p}', t) \leqslant 1+\operatorname{rank}\mathfrak{p}' \leqslant p-m+2.$$

But $(\mathfrak{a}, t) \subseteq (\mathfrak{b}', t) \subseteq (\mathfrak{p}', t)$, so that $(\mathfrak{a}, t)$ has rank not exceeding $p-m+2$. But the rank of $\mathfrak{a}R[t]$ is strictly less than that of $(\mathfrak{a}, t)$ because t is not contained in any prime ideal of $\mathfrak{a}R[t]$, while $\mathfrak{a}$ and $\mathfrak{a}R[t]$ have the same rank [(**5**) 67 Prop. 5]. Combining these remarks we obtain

$$\operatorname{rank}\mathfrak{a} \leqslant p-m+1,$$

and this completes the proof.

I conclude this section with some further remarks on the use of indeterminates. Let x_{ij} $(1 \leqslant i, j \leqslant p)$ be indeterminates, put

$$\alpha_{ij} = \sum_{\mu=1}^{p} a_{i\mu} x_{\mu j} \qquad \text{(G)}$$

and, with the notation of § 3, write $P = R_{(\ldots x_{ij} \ldots)}$. Then in P the matrices $\|\alpha_{ij}\|$ and $\|a_{ij}\|$ determine the same ideal, this being a proper P-ideal if (F) is a proper R-ideal. When R is Noetherian, the P-ideal and R-ideal have similar primary decompositions. In particular, the ideal (F) and its extension to P have the same rank.

† Here we make use of Nagata's theorem and also the corollary to Theorem 1.

LEMMA 3. *If R is semi-regular, $m \leqslant n \leqslant k+m-1$, and* (F) *is a proper R-ideal of rank k, then any n columns of the matrix $\|\alpha_{ij}\|$ determine a P-ideal of rank $n-m+1$.*

The proof involves only simple adjustments to that given in § 3 in connexion with a similar but less complex situation. However, we need semi-regularity here in order to assert that a proper ideal defined by a matrix with s rows and t columns ($s \leqslant t$) has rank not exceeding $t-s+1$.

COROLLARY. *Let R be semi-regular and suppose that* (F) *is a proper ideal of rank $p-m+1$. If now $p > m$, then*

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \alpha_{1,p-1} \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \alpha_{2,p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m1} & \alpha_{m2} & \cdot & \cdot & \alpha_{m,p-1} \end{pmatrix}$$

has rank $p-m$, while, if $p > m \geqslant 2$, then the ideal

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \alpha_{1,p-1} \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \alpha_{2,p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m-1,1} & \alpha_{m-1,2} & \cdot & \cdot & \alpha_{m-1,p-1} \end{pmatrix}, \tag{H}$$

provided it is proper, has rank $p-m+1$.

Proof. The first assertion follows at once from the lemma. Now suppose that the first $m-1$ rows of (F) define a proper R-ideal; then its rank k' is at least equal to that of (F) and consequently $k' \geqslant p-m+1$. Thus

$$(m-1) < (p-1) \leqslant k'+(m-1)-1,$$

and so, again by the lemma, (H) has rank $p-m+1$. Finally, consider the case in which the ideal determined by the first $m-1$ rows of (F) is the whole of R. In this situation every $(m-1)$-rowed minor of

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \alpha_{1p} \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \alpha_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m-1,1} & \alpha_{m-1,2} & \cdot & \cdot & \alpha_{m-1,p} \end{pmatrix}$$

is a unit in P, and so, in particular, (H) is not proper.

6. Ideals defined by matrices (continued)

In this concluding section, we consider a semi-regular ring R and a proper R-ideal $\mathfrak{a}$ defined by a matrix $\|a_{ij}\|$ with m rows and p columns, where $m \leqslant p$. Furthermore, we shall be concerned with the case in

which the rank† of $\mathfrak{a}$ is $p-m+1$ (the maximum value possible) and our results can be regarded as extending known theorems on ideals of the principal class. For polynomial rings, these extensions are due to Macaulay [see (**3**) §§ 53, 93]. The proofs given here follow the original arguments of Macaulay very closely, for we have reached a point where the changes involved in working with semi-regular rather than polynomial rings are comparatively minor.

To simplify the statement of Theorem 10, we put

$$A_{s_1 s_2 \ldots s_m} = \begin{vmatrix} a_{1s_1} & a_{1s_2} & . & . & a_{1s_m} \\ a_{2s_1} & a_{2s_2} & . & . & a_{2s_m} \\ . & . & . & . & . & . \\ a_{ms_1} & a_{ms_2} & . & . & a_{ms_m} \end{vmatrix}$$

Naturally, if any two of $s_1, s_2, \ldots, s_m$ are equal, then $A_{s_1 s_2 \ldots s_m} = 0$; while, if they are all different and we subject them to a permutation, then the effect is to multiply $A_{s_1 s_2 \ldots s_m}$ by $+1$ or -1 according as the permutation is even or odd. This will be expressed by saying that $A_{s_1 s_2 \ldots s_m}$ *is an alternating function of its suffixes*, and the same expression will be used for other algebraic entities which exhibit a similar behaviour in regard to their suffixes.

THEOREM 10. *Let R be a semi-regular ring and*

$$\mathfrak{a} = \begin{pmatrix} a_{11} & a_{12} & . & . & a_{1p} \\ a_{21} & a_{22} & . & . & a_{2p} \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mp} \end{pmatrix} \quad (m \leqslant p)$$

a proper ideal of rank $p-m+1$. Suppose further that (*with the above notation*)

$$\sum_{s_1 < s_2 < \ldots < s_m} A_{s_1 s_2 \ldots s_m} L_{s_1 s_2 \ldots s_m} = 0, \tag{I}$$

where the $L_{s_1 s_2 \ldots s_m}$ $(1 \leqslant s_1 < s_2 < \ldots < s_m \leqslant p)$ are elements of R. Then

$$L_{s_1 s_2 \ldots s_m} = \sum_{s=1}^{p} D^{(1)}_{s_1 s_2 \ldots s_m s} a_{1s} + \sum_{s=1}^{p} D^{(2)}_{s_1 s_2 \ldots s_m s} a_{2s} + \ldots + \sum_{s=1}^{p} D^{(m)}_{s_1 s_2 \ldots s_m s} a_{ms}, \tag{J}$$

where, for $1 \leqslant t \leqslant m$, $D^{(t)}_{s_1 s_2 \ldots s_m s}$ belongs to R and is an alternating function of its suffixes.

Remark. The converse of the theorem holds for relatively trivial reasons. In other words, if elements $D^{(t)}_{s_1 s_2 \ldots s_{m+1}}$ $(1 \leqslant t \leqslant m)$ are alternating functions of their suffixes and $L_{s_1 s_2 \ldots s_m}$ is defined by (J), then (I)

† We mean, of course, the rank of $\mathfrak{a}$ as an ideal and not the rank of the underlying matrix.

will be true. This follows from the fact that a determinant is zero when it has two identical rows.

THEOREM 11. *Let R be a semi-regular ring and*

$$\mathfrak{a} = \begin{pmatrix} a_{11} & a_{12} & . & . & a_{1p} \\ a_{21} & a_{22} & . & . & a_{2p} \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mp} \end{pmatrix} \quad (m \leqslant p)$$

a proper ideal of rank $p-m+1$. Then $\mathfrak{a}$ is semi-regular.

Combined proofs of Theorems 10 *and* 11. Following Macaulay we shall prove Theorems 10 and 11 simultaneously using an inductive argument. Observe first that, when $m = 1$, they reduce to Theorems 8 and 6 respectively. Accordingly we shall suppose that $m \geqslant 2$ and that both theorems have been established for ideals derived from matrices with a smaller number of rows. Next we observe that, when $p = m$, both theorems are obvious. Accordingly we shall also suppose that $p > m$ and that the theorems are known to hold for every ideal defined by an $m \times h$ matrix, provided that $m \leqslant h < p$. The inductive step consists in extending the two theorems to the case of $m \times p$ matrices.

The inductive step (*Theorem* 10). We now deduce Theorem 10 for an ideal $\mathfrak{a}$ defined by an $m \times p$ matrix, but, as preparation, we require a number of observations of a rather general character. Observe, to begin with, that, if

$$a'_{ij} = \sum_{\mu=1}^{p} a_{i\mu} c_{\mu j} \quad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant p),$$

where the determinant $|c_{\mu j}|$ is a unit of the ring R, then

$$\begin{pmatrix} a'_{11} & a'_{12} & . & . & a'_{1p} \\ a'_{21} & a'_{22} & . & . & a'_{2p} \\ . & . & . & . & . \\ a'_{m1} & a'_{m2} & . & . & a'_{mp} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & . & . & a_{1p} \\ a_{21} & a_{22} & . & . & a_{2p} \\ . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{mp} \end{pmatrix}.$$

Moreover, if we can establish Theorem 10 for the former, then it will follow for the latter. In particular, we may, if we wish, transform the given matrix by any number of operations which involve adding a multiple (by an element of R) of one column to a different one.

Again, suppose that we put

$$\alpha_{ij} = \sum_{\mu=1}^{p} a_{i\mu} x_{\mu j},$$

where the $x_{\mu j}$ are indeterminates, and [see § 3] pass to the ring $P = R_{(\ldots x_{ij} \ldots)}$. Then, since the determinant $|x_{\mu j}|$ is a unit in P, it follows readily from the above remarks that it will suffice to prove Theorem 10

for the P-ideal determined by the matrix $\|\alpha_{ij}\|$. This ideal is also of rank $p-m+1$. Furthermore, since $p > m$,

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \alpha_{1,p-1} \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \alpha_{2,p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m1} & \alpha_{m2} & \cdot & \cdot & \alpha_{m,p-1} \end{pmatrix}$$

has rank $p-m$ by the Corollary to Lemma 3.

Next observe that, if the R-ideal generated by $a_{11}, a_{12}, \ldots, a_{1p}$ is the whole of R, then each of $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1p}$ is a unit of P. In this special situation, the matrix $\|\alpha_{ij}\|$ can be transformed, by elementary column operations, to one of the form

$$\begin{Vmatrix} \alpha_{11} & 0 & 0 & \cdot & \cdot & 0 \\ \alpha'_{21} & \alpha'_{22} & \alpha'_{23} & \cdot & \cdot & \alpha'_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha'_{m1} & \alpha'_{m2} & \alpha'_{m3} & \cdot & \cdot & \alpha'_{mp} \end{Vmatrix}.$$

However, for such a case, the required result follows by applying Theorem 10 to the $(m-1)\times(p-1)$ matrix one obtains by striking out the first row and column. Accordingly, we may suppose that $a_{11}, a_{12}, \ldots, a_{1p}$ generate a proper R-ideal, and then

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \alpha_{1,p-1} \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \alpha_{2,p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m-1,1} & \alpha_{m-1,2} & \cdot & \cdot & \alpha_{m-1,p-1} \end{pmatrix}$$

is a proper P-ideal. The rank of this ideal is $p-m+1$ by virtue of the corollary to Lemma 3.

Recapitulation. We can sum up the above remarks as follows. When establishing the inductive step for Theorem 10, we are free to assume (in addition to what is given) that

$$\mathfrak{a}_1 = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1,p-1} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2,p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{m,p-1} \end{pmatrix}$$

has rank $p-m$ and

$$\mathfrak{a}'_1 = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1,p-1} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2,p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m-1,1} & a_{m-1,2} & \cdot & \cdot & a_{m-1,p-1} \end{pmatrix}$$

has rank $p-m+1$. However, one further preparatory step is necessary

which is to arrange that

$$\mathfrak{a}_1 : A_{12\ldots(m-1)p} = \mathfrak{a}_1.$$

This can be achieved by adding to the rth ($1 \leqslant r \leqslant m-1$) column, of the matrix defining $\mathfrak{a}$, a linear combination of the $(r+1)$th, $(r+2)$th,..., $(p-1)$th columns, the coefficients being indeterminates which are adjoined to R but without introducing new denominators. This modifies the matrices associated with $\mathfrak{a}_1$ and $\mathfrak{a}_1'$ as well (but without destroying the properties already secured for them), and it will still suffice to prove Theorem 10 for the new $\mathfrak{a}$-matrix. Further, the additional requirement $\mathfrak{a}_1 : A_{12\ldots(m-1)p} = \mathfrak{a}_1$ is now satisfied. For otherwise there would be a prime ideal $\mathfrak{p}$, belonging to the extension of $\mathfrak{a}_1$, containing the new $A_{12\ldots(m-1)p}$. The rank of $\mathfrak{p}$ would be $p-m$ (inductive hypothesis for Theorem 11) and $\mathfrak{p}$ itself would be the extension of a prime R-ideal. Accordingly, all *original* determinants

$$A_{s_1 s_2 \ldots s_{m-1} p} \quad (s_1 < \ldots < s_{m-1} < p)$$

would be contained in $\mathfrak{p}$, and so we should have $\mathfrak{a} \subseteq \mathfrak{p}$. This, however, contradicts the hypothesis concerning the rank of $\mathfrak{a}$.

After these general considerations we come to the computational aspects of Macaulay's argument. Put

$$P_r = (-1)^{m+r} \begin{vmatrix} a_{11} & a_{12} & . & . & a_{1,m-1} \\ . & . & . & . & . & . & . \\ \hat{a}_{r1} & \hat{a}_{r2} & . & . & \hat{a}_{r,m-1} \\ . & . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & a_{m,m-1} \end{vmatrix},$$

where the symbol $\hat{}$ over the elements of the rth row indicate that that particular row is to be omitted. If now $s_1 < s_2 < \ldots < s_m$, then

$$a_{1s_i} P_1 + a_{2s_i} P_2 + \ldots + a_{ms_i} P_m = A_{12\ldots(m-1)s_i},$$

whence, eliminating $P_1, P_2, \ldots, P_{m-1}$, we have

$$P_m A_{s_1 s_2 \ldots s_m} = \sum_{r=1}^{m} (-1)^{m+r} B_{s_1 \ldots \hat{s}_r \ldots s_m} A_{12\ldots(m-1)s_r},$$

where

$$B_{\sigma_1 \sigma_2 \ldots \sigma_{m-1}} = \begin{vmatrix} a_{1\sigma_1} & a_{1\sigma_2} & . & . & a_{1\sigma_{m-1}} \\ a_{2\sigma_1} & a_{2\sigma_2} & . & . & a_{2\sigma_{m-1}} \\ . & . & . & . & . & . \\ a_{m-1,\sigma_1} & a_{m-1\,\sigma_2} & . & . & a_{m-1,\sigma_{m-1}} \end{vmatrix}.$$

Multiplying the relation

$$\sum_{s_1 < s_2 < \ldots < s_m} A_{s_1 s_2 \ldots s_m} l_{s_1 s_2 \ldots s_m} = 0$$

by P_m and using the above expression, we get

$$\sum_{s_1<s_2<\ldots<s_m} L_{s_1 s_2 \ldots s_m}\Big(\sum_{r=1}^{m}(-1)^{m+r}B_{s_1\ldots\hat{s}_r\ldots s_m}A_{12\ldots(m-1)s_r}\Big)=0,$$

and, if we pick out the coefficient of $A_{12\ldots(m-1)p}$, we find that

$$A_{12\ldots(m-1)p}\sum_{s_1<s_2<\ldots<s_{m-1}<p} L_{s_1 s_2\ldots s_{m-1}p}B_{s_1 s_2\ldots s_{m-1}}\equiv 0 \pmod{\mathfrak{a}_1}.$$

But $\mathfrak{a}:A_{12\ldots(m-1)p}=\mathfrak{a}_1$, and so

$$\sum_{s_1<\ldots<s_{m-1}<p} L_{s_1 s_2\ldots s_{m-1}p}B_{s_1 s_2\ldots s_{m-1}}=\sum_{t_1<t_2<\ldots<t_m<p} A_{t_1 t_2\ldots t_m}\Delta_{t_1 t_2\ldots t_m},$$

where the $\Delta_{t_1 t_2\ldots t_m}$ are elements of R. Furthermore

$$A_{t_1 t_2\ldots t_m}=\sum_{r=1}^{m}(-1)^{m+r}a_{mt_r}B_{t_1\ldots\hat{t}_r\ldots t_m},$$

which on substitution gives

$$\sum_{s_1<\ldots<s_{m-1}<p} L_{s_1 s_2\ldots s_{m-1}p}B_{s_1 s_2\ldots s_{m-1}}$$
$$=\sum_{s_1<\ldots<s_{m-1}<p} B_{s_1 s_2\ldots s_{m-1}}\Big(\sum_{s=1}^{p-1}\Delta_{s_1\ldots s_{m-1}s}a_{ms}\Big),$$

where $\Delta_{s_1\ldots s_{m-1}s}$ is to be evaluated by extending the definition of $\Delta_{t_1 t_2\ldots t_m}$ so as to make it an alternating function of its suffixes. Now the $B_{s_1 s_2\ldots s_{m-1}}$ are the $(m-1)$-rowed minors of the matrix associated with $\mathfrak{a}_1'$; consequently we can apply Theorem 10 (by virtue of the inductive hypothesis) to obtain

$$L_{s_1\ldots s_{m-1}p}=\sum_{s=1}^{p-1}\Delta^{(1)}_{s_1\ldots s_{m-1}s}a_{1s}+\ldots+\sum_{s=1}^{p-1}\Delta^{(m-1)}_{s_1\ldots s_{m-1}s}a_{m-1,s}+\sum_{s=1}^{p-1}\Delta^{(m)}_{s_1\ldots s_{m-1}s}a_{ms},$$

where $s_1<s_2<\ldots<s_{m-1}<p$, each of $\Delta^{(1)}_{s_1\ldots s_{m-1}s},\ldots,\Delta^{(m-1)}_{s_1\ldots s_{m-1}s}$ is an alternating function of its suffixes, and

$$\Delta^{(m)}_{s_1\ldots s_{m-1}s}=\Delta_{s_1\ldots s_{m-1}s}.$$

Let us now use the above expression for $L_{s_1\ldots s_{m-1}p}$ to calculate

$$\sum_{s_1<\ldots<s_{m-1}<p} A_{s_1\ldots s_{m-1}p}L_{s_1\ldots s_{m-1}p}.$$

If $t_1<t_2<\ldots<t_m<p$, then the term involving multiples of $\Delta^{(q)}_{t_1 t_2\ldots t_m}$ is

$$\sum_{r=1}^{m}\Delta^{(q)}_{t_1\ldots\hat{t}_r\ldots t_m t_r}a_{qt_r}A_{t_1\ldots\hat{t}_r\ldots t_m p}=\Delta^{(q)}_{t_1 t_2\ldots t_m}\sum_{r=1}^{m}(-1)^{m+r}a_{qt_r}A_{t_1\ldots\hat{t}_r\ldots t_m p}$$
$$=-\Delta^{(q)}_{t_1 t_2\ldots t_m}\begin{vmatrix} a_{1t_1} & a_{1t_2} & \cdot & \cdot & a_{1t_m} & a_{1p}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ a_{mt_1} & a_{mt_2} & \cdot & \cdot & a_{mt_m} & a_{mp}\\ a_{qt_1} & a_{qt_2} & \cdot & \cdot & a_{qt_m} & 0 \end{vmatrix}$$
$$=A_{t_1 t_2\ldots t_m}\Delta^{(q)}_{t_1 t_2\ldots t_m}a_{qp},$$

so that

$$\sum_{s_1<\ldots<s_{m-1}<p} A_{s_1\ldots s_{m-1}p} L_{s_1\ldots s_{m-1}p} = \sum_{t_1<\ldots<t_m<p} A_{t_1t_2\ldots t_m}(\Delta^{(1)}_{t_1t_2\ldots t_m} a_{1p}+\ldots+\Delta^{(m)}_{t_1t_2\ldots t_m} a_{mp}).$$

But

$$\sum_{s_1<\ldots<s_m} A_{s_1\ldots s_m} L_{s_1\ldots s_m} = 0,$$

and consequently we have

$$\sum_{t_1<t_2<\ldots<t_m<p} A_{t_1t_2\ldots t_m}(L_{t_1t_2\ldots t_m}+\Delta^{(1)}_{t_1t_2\ldots t_m} a_{1p}+\ldots+\Delta^{(m)}_{t_1t_2\ldots t_m} a_{mp}) = 0,$$

and therefore, by the inductive hypothesis (applying Theorem 10 to $\mathfrak{a}_1$),

$$L_{t_1t_2\ldots t_m}+\Delta^{(1)}_{t_1t_2\ldots t_m} a_{1p}+\ldots+\Delta^{(m)}_{t_1t_2\ldots t_m} a_{mp} = \sum_{s=1}^{p-1} P^{(1)}_{t_1t_2\ldots t_ms} a_{1s}+\ldots+\sum_{s=1}^{p-1} P^{(m)}_{t_1t_2\ldots t_ms} a_{ms},$$

where $t_1 < t_2 < \ldots < t_m < p$ and the $P^{(q)}_{t_1t_2\ldots t_ms}$ have the usual alternating properties. Finally, if we set

$$D^{(q)}_{s_1s_2\ldots s_mp} = -\Delta^{(q)}_{s_1s_2\ldots s_m} \qquad (s_1 < \ldots < s_m < p),$$

$$D^{(q)}_{s_1s_2\ldots s_ms_{m+1}} = P^{(q)}_{s_1s_2\ldots s_ms_{m+1}} \qquad (s_1 < s_2 < \ldots < s_{m+1} < p)$$

and extend by making the elements $D^{(q)}$ alternating functions of their suffixes, then

$$L_{s_1s_2\ldots s_m} = \sum_{s=1}^{p} D^{(1)}_{s_1s_2\ldots s_ms} a_{1s}+\ldots+\sum_{s=1}^{p} D^{(m)}_{s_1s_2\ldots s_ms} a_{ms}$$

whenever $s_1 < s_2 < \ldots < s_m \leqslant p$. This completes the inductive step in the case of Theorem 10.

The inductive step (*Theorem* 11). It remains for us to derive Theorem 11 for ideals defined by $m \times p$ matrices and for this (by Theorem 1) we need consider only the case in which R is a semi-regular local ring. Accordingly, it will be assumed that R has a single maximal ideal $\mathfrak{m}$ and that the dimension of R, as a local ring, is d.

With the previous notation, put $k = \operatorname{rank} \mathfrak{a}$, so that $k = p-m+1$. Then $k \leqslant d$, and it will suffice to show that there exist non-units $\gamma_1, \gamma_2, \ldots, \gamma_{d-k}$ such that

$$(\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_{i-1}) : \gamma_i = (\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_{i-1})$$

for $i = 1, 2, \ldots, d-k$. These elements will be constructed in succession, and it will be convenient to make them satisfy the additional requirement that

$$(\gamma_1, \gamma_2, \ldots, \gamma_{i-1}) : \gamma_i = (\gamma_1, \gamma_2, \ldots, \gamma_{i-1}) \quad (1 \leqslant i \leqslant d-k).$$

Suppose now that $0 \leqslant q < d-k$ and that $\gamma_1, \gamma_2, \ldots, \gamma_q$ have been constructed in accordance with the above specifications. Using Theorem 5 and known properties of semi-regular local rings [(7) 372 Theorem 2.3] we see that $(\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_q)$ has rank $k+q$ and dimension $d-k-q$. Choose a non-unit γ_{q+1}, not contained in any minimal prime ideal of $(\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_q)$, such that

$$(\gamma_1, \gamma_2, \ldots, \gamma_q) : \gamma_{q+1} = (\gamma_1, \gamma_2, \ldots, \gamma_q).$$

We contend that

$$(\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_q) : \gamma_{q+1} = (\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_q).$$

To establish this, suppose that $\gamma_{q+1} x \in (\mathfrak{a}, \gamma_1, \ldots, \gamma_q)$ then [cf. Macaulay (3) 57]

$$\gamma_{q+1} x = \sum_{s_1 < s_2 < \ldots < s_m} A_{s_1 s_2 \ldots s_m} L_{s_1 s_2 \ldots s_m} + u_1 \gamma_1 + \ldots + u_q \gamma_q, \qquad \text{(K)}$$

where the $L_{s_1 s_2 \ldots s_m}$ and $u_1, u_2, \ldots, u_q$ all belong to R.

Now pass to $R/(\gamma_1, \gamma_2, \ldots, \gamma_{q+1})$, which, it should be noted, is a semi-regular local ring of dimension $d-q-1$, and use an asterisk to indicate the effect of applying the natural mapping $R \to R/(\gamma_1, \gamma_2, \ldots, \gamma_{q+1})$. This gives

$$\sum_{s_1 < s_2 < \ldots < s_m} A^*_{s_1 s_2 \ldots s_m} L^*_{s_1 s_2 \ldots s_m} = 0.$$

But $\mathfrak{a}^*$ has dimension $d-k-q-1$ and therefore rank $\mathfrak{a}^* = k = p-m+1$; consequently we are in a position to apply Theorem 10 to $\mathfrak{a}^*$ regarded as an ideal defined by an $m \times p$ matrix. Accordingly

$$L^*_{s_1 s_2 \ldots s_m} = \sum_{s=1}^{p} D^{(1)*}_{s_1 s_2 \ldots s_m s} a^*_{1s} + \ldots + \sum_{s=1}^{p} D^{(m)*}_{s_1 s_2 \ldots s_m s} a^*_{ms},$$

where $D^{(t)}_{s_1 s_2 \ldots s_m s}$ belongs to R and is an alternating function of its suffixes. Thus

$$L_{s_1 s_2 \ldots s_m} \equiv \sum_{s=1}^{p} D^{(1)}_{s_1 s_2 \ldots s_m s} a_{1s} + \ldots + \sum_{s=1}^{p} D^{(m)}_{s_1 s_2 \ldots s_m s} a_{ms}$$

modulo $(\gamma_1, \gamma_2, \ldots, \gamma_{q+1})$. Hence, by substituting in (K), we get

$$\gamma_{q+1} x \equiv \gamma_{q+1} \alpha \pmod{(\gamma_1, \gamma_2, \ldots, \gamma_q)}$$

where $\alpha \in \mathfrak{a}$. But

$$(\gamma_1, \gamma_2, \ldots, \gamma_q) : \gamma_{q+1} = (\gamma_1, \gamma_2, \ldots, \gamma_q).$$

Hence $(x-\alpha) \in (\gamma_1, \gamma_2, \ldots, \gamma_q)$ and therefore $x \in (\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_q)$. This shows that

$$(\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_q) : \gamma_{q+1} = (\mathfrak{a}, \gamma_1, \gamma_2, \ldots, \gamma_q)$$

and proves that it is always possible to continue the sequence $\gamma_1, \gamma_2, \ldots$ until the full set of $d-k$ terms has been obtained.

The demonstration of Theorems 10 and 11 is now complete.

REFERENCES

1. I. S. Cohen, 'On the structure and ideal theory of complete local rings', *Trans. American Math. Soc.* 59 (1946) 54–106.
2. W. Gröbner, *Moderne algebraische Geometrie* (Berlin, 1949).
3. F. S. Macaulay, *The algebraic theory of modular systems*, Cambridge Tracts, 19 (1916).
4. M. Nagata, *The theory of multiplicity in general local rings*, International Symposium on Algebraic Number Theory (Tokyo and Nikko, 1955), 191–236.
5. D. G. Northcott, *Ideal theory*, Cambridge Tracts, 42 (1953).
6. —— 'Semi-regular local rings', *Mathematika* 3 (1956) 117–26.
7. D. G. Northcott and D. Rees, 'Extensions and simplifications of the theory of regular local rings', *J. London Math. Soc.* 32 (1957) 367–74.
8. D. Rees, 'The grade of an ideal or module', *Proc. Cambridge Phil. Soc.* 53 (1957) 28–42.
9. O. Zariski and P. Samuel, *Commutative algebra*, vol. I (Princeton, 1958).

UNIVERSAL SPACES FOR SOME METRIZABLE UNIFORMITIES

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1. Introduction

A *uniformity* on a space R is here regarded, following Tukey (**10**), as a suitable family of 'uniform' coverings of R ('large' coverings in the notation of (**10**)). It is well known that, if a metrizable uniformity on R has a basis of *finite* coverings, then R is separable and the metric ρ is totally bounded, and conversely; equivalently, the uniformity (R, ρ) is *uniformly* imbeddable in the Hilbert cube.† Again, if a metrizable uniformity on R has a basis of *countable* coverings, then R is separable, and conversely (the metric can here be arbitrary); equivalently, the uniformity is uniformly imbeddable in the Banach space of continuous functions on the unit interval.‡ These examples suggest that other metrizable uniformities, definable in terms of simple kinds of coverings, may also be characterized by their (uniform) imbeddability in 'universal' spaces of the same kind. I shall here prove some results of this type. There may well be others, which should be of interest since relatively little is known about uniform (as opposed to topological) imbedding.

2. Notation

The uniformities and metrics on a space R are to agree with the topology of R, and the metrics ρ for a metrizable uniform space are to agree with the uniformity. We use the notation (R, ρ) both for a metric space and for the corresponding uniformity, but often omit the ρ when the uniformity intended is clear. It is understood that a product of (finitely many or $\aleph_0$) metric spaces is to be given a metric agreeing with the product uniformity, i.e. either the usual product metric [e.g. as in (**5**) 105–6], or one uniformly equivalent to it. A subspace X of a metric space (Y, σ) is understood to inherit the metric σ.

All coverings considered are to be *open*. A covering $\mathscr{U}$ is *star-finite*,

† By a *uniform imbedding* of a uniform space X in a uniform space Y, we mean a 1–1 mapping f of X in Y such that both f and f^{-1} are uniformly continuous: that is, f is a uniform equivalence.

‡ See (**2**) 187; the imbedding can in fact be made isometric.

Quart. J. Math. Oxford (2), 11 (1960), 105–15.

or *star-bounded*, or *star-countable* if each U in $\mathscr{U}$ meets only a finite, or bounded, or countable, number of sets of $\mathscr{U}$ respectively. An 'ϵ-covering' of (R, ρ) is one by sets of diameters less than ϵ. The 'ϵ-sphere' $S(x, \epsilon)$ is $\{y \mid y \in R, \rho(x, y) < \epsilon\}$.

We use H for the Hilbert cube, E^m and $E^{\aleph_0}$ for the products of m and $\aleph_0$ copies of the real line E^1 (with usual metric), and C for the Banach space of continuous real-valued functions on the unit interval (normed by $\|f\| = \max|f(t)|$). Finally, B denotes a 'generalized Baire space' in the sense of Morita [(6) 361]; that is, B is the product of $\aleph_0$ infinite discrete spaces D_1, D_2,..., where the distance between distinct points of each D_n is 1. A convenient metric for B (uniformly equivalent to the product metric) is given by $\rho(\{x_n\}, \{y_n\}) = 1/n$ when $x_1 = y_1$,..., $x_{n-1} = y_{n-1}$, $x_n \neq y_n$. For our purposes nothing is lost by requiring the sets D_n all to have the same cardinal.

In what follows, all complexes are supposed to be realized geometrically as metric spaces, the metric being the Hilbert space metric on the barycentric coordinates.

3. N-spaces

We say that a metric space (R, ρ) (or, more accurately, the corresponding metrizable uniformity on R) is an '*N-space*' if it has the property that every covering of R is uniform. J. Nagata (7) has shown that it suffices to require that every *finite* covering is uniform, and that the N-spaces are precisely those for which the topology satisfies

[3.1] *The set R' of limit points of R is compact,*

and for which the metric satisfies

[3.2] *To each $\epsilon > 0$ corresponds $\delta > 0$ such that, if $0 < \rho(x, y) < \delta$, then $\rho(x, R') < \epsilon$ and $\rho(y, R') < \epsilon$.*†

An alternative proof is furnished by (1) [in view of (9) 124, Theorem X], where further characterizations are given. It is easily deduced from [3.1] and [3.2] that every covering of R has a star-bounded refinement, so that an N-space can also be characterized as a metrizable uniformity having the family of *all star-bounded coverings* (or any larger family of coverings) as a basis.

THEOREM 1. *For each infinite cardinal number* $\mathfrak{m}$, *there exists a metric space* (H^*, σ^*) *such that* (*a*) H^* *and all its closed subspaces are N-spaces*

† As is shown in (7) [3.1] is the only condition on the topology of R. In fact, every metrizable space R with $R' \neq \emptyset$ has a metric satisfying [3.2] in the stronger form

$$\rho(x, y) \geq \rho(x, R') + \rho(y, R') \quad (\text{if } x \neq y).$$

and have dense subsets of cardinal $\leqslant \mathfrak{m}$, (*b*) *conversely, every N-space with a dense subset of cardinal* $\leqslant \mathfrak{m}$ *is uniformly imbeddable as a closed subspace of* H^*.

Proof. Let A be a dense subset of (H, σ), the Hilbert cube with its usual metric. For each a in A and each positive integer n, take a set $E(a, n)$ of $\mathfrak{m}$ points; all the sets $E(a, n)$ are to be disjoint from each other and from H. Let

$$H^* = H \cup \bigcup\{E(a, n) \mid a \in A \ (n = 1, 2, \ldots)\},$$

and define σ^* as follows. If $y_1, y_2 \in H$,

$$\sigma^*(y_1, y_2) = \sigma(y_1, y_2).$$

If $y_1 \in E(a, n)$ and $y_2 \in H$,

$$\sigma^*(y_1, y_2) = \sigma(a, y_1) + 1/n.$$

If $y_1 \in E(a, n)$ and $y_2 \in E(b, m)$ (where $y_1 \neq y_2$ but m, n or a, b may coincide)

$$\sigma^*(y_1, y_2) = \sigma(a, b) + 1/n + 1/m.$$

It is easily verified that σ^* is a metric on H^*, reducing to σ on H. Further, $(H^*)' = H$ and is compact, and condition [3.2] is also satisfied, and so (H^*, σ^*) is an N-space; and the set $\bigcup E(a, n)$ is dense and of cardinal $\mathfrak{m}$. It follows (since every covering is uniform) that every closed subspace of H^* (with metric σ^*) has the properties stated in (*a*) of the theorem.

Conversely, let (R, ρ) be an N-space with a dense set B of cardinal $\leqslant \mathfrak{m}$. By [3.1], R' is compact, and there is a homeomorphism f of R' in H; further, $R - R' \subset B$ and so has $\leqslant \mathfrak{m}$ points. We extend f to a mapping f^* of R in H^* as follows. If $R' = \emptyset$, f^* is simply an arbitrary (1–1) mapping of R in any one $E(a, n)$. Suppose, then, that $R' \neq \emptyset$, and let $x \in R - R'$ be given. Choose $z = z(x) \in R'$, so that $\rho(x, z)$ has its least value $\rho(x, R')$, and take $n = n(x)$ to be the smallest positive integer for which $1/n < \rho(x, z(x))$. Pick $a = a(x) \in A$, so that $\sigma(a, f(z)) < 1/n(x)$. There is a (1–1) mapping $\phi_{a,n}$ of $R - R'$ onto a subset of $E(a, n)$. Define

$$f^*(x) = \phi_{a(x), n(x)}(x)$$

for x in $R - R'$, and

$$f^*(x) = f(x)$$

for x in R'. Clearly f^* is (1–1), and $f^*(R)$ is closed in H^*; and it is easy to check that f^* and $(f^*)^{-1}$ are continuous. Since the homeomorphism f^* takes the uniformity of R (of all coverings) into that of $f^*(R)$, f^* is a uniform equivalence (as is also easily seen directly).

Remark. The case $\mathfrak{m} = \aleph_0$ of this theorem gives the existence of a

'universal' space for the metrizable uniformities (R,ρ) which have the family of *all countable coverings* of R as a basis. For these are simply the separable N-spaces. A similar remark applies to general $\mathfrak{m}$. There is also a similar (but simpler) universal-space theorem for the metrizable uniformities (R,ρ) for which the family of *all finite coverings* of R is a basis. For any such (R,ρ) must be compact (as follows easily from the fact that it must be an N-space), and so imbeddable (uniformly) as a closed subspace in H (and conversely).

4. Euclidean uniformities

We say that a metrizable uniformity (R,ρ) is 'Euclidean' if it has a basis of 'Euclidean coverings' in the sense of J. R. Isbell [(**4**) 70]; here a Euclidean covering $\mathscr{U}$ of R is one for which the (metric) nerve N of $\mathscr{U}$ is uniformly imbeddable in some Euclidean space E^m (m depending on $\mathscr{U}$). The following theorem is essentially a reformulation of part of [(**4**) 1.7] in a special case.

THEOREM 2. *A metrizable uniformity is Euclidean if and only if it can be uniformly imbedded in E^∞.*

We first observe that every Euclidean covering $\mathscr{U}$ of (R,ρ) has a refinement of the form

$$\mathscr{V} = f^{-1}(\mathscr{W}), \tag{1}$$

where $\mathscr{W}$ is a uniform covering of some Euclidean space E^m, and f is a uniformly continuous mapping of R in E^m.

For, as shown in (**4**), since $\mathscr{U}$ is finite-dimensional, there is a uniformly continuous mapping f of R in the nerve N of $\mathscr{U}$, where we may assume that $N \subset E^m$; and a suitable $\mathscr{W}$ is easily found. Conversely, every covering of the form (1) has a Euclidean refinement, obtained by applying f^{-1} to the covering by (open) stars of vertices in a suitable simplicial subdivision of E^m. If now (R,ρ) is Euclidean, there is for each n ($n = 1, 2, \ldots$) a uniform $(1/n)$-covering $\mathscr{V}_n = f_n^{-1}(\mathscr{W}_n)$, as in (1), where $\mathscr{W}_n$ is a uniform covering of $E^{m(n)}$. Let f be the product mapping

$$f(x) = \{f_1(x), f_2(x), \ldots\}$$

of R in $$\prod \{E^{m(n)} \mid n = 1, 2, \ldots\};$$

it is easily seen that f is a uniform equivalence between R and $f(R)$, and, since

$$\prod E^{m(n)} = \prod (E^1 \times E^1 \times \ldots \times E^1) = E^\infty,$$

this gives the desired imbedding. Conversely, E^∞ (or any subspace of it) has a basis of uniform coverings of the form (1), and so is Euclidean.

A similar argument applies to the (not necessarily metrizable) uniformities on a space R which have a sub-basis of coverings of the form

(1), but with the Euclidean spaces E^m replaced by an arbitrary family of uniform spaces E_λ. Such uniformities are characterized as being uniformly imbeddable in $\prod_\lambda (E_\lambda)^{\mathfrak{m}}$, $\mathfrak{m}$ being the least cardinal of a basis for the uniformity on R. The metrizable uniformities on R correspond to $\mathfrak{m} \leqslant \aleph_0$.

5. M-spaces

We say that a metrizable uniformity (R, ρ) is (a) an *M_b-space*, (b) an *M_f-space*, or (c) an *M_c-space*, if its uniformity has a basis of (a) star-bounded, (b) star-finite, (c) star-countable coverings, respectively. The following theorem shows the existence of 'universal' M_c-spaces.

THEOREM 3. *For each infinite cardinal number $\mathfrak{m}$ there is a metric space $(C \times B, \tau)$, of which every subspace is an M_c-space with a dense subset of $\leqslant \mathfrak{m}$ points, such that every M_c-space with a dense subset of $\leqslant \mathfrak{m}$ points is uniformly imbeddable in $(C \times B, \tau)$.*

We take B to be the generalized Baire space $\prod D_n$, where each discrete space D_n has cardinal $\mathfrak{m}$; C is the Banach space described in § 2, and τ is the usual product metric. Since C is separable, $C \times B$ has a dense subset of $\mathfrak{m}$ points. Further, for each $\epsilon > 0$, we obtain a star-countable uniform ϵ-covering of $C \times B$ as follows. Take a countable $\frac{1}{4}\epsilon$-covering $\{U_k\}$ $(k = 1, 2, \ldots)$ of C; let V_k be the $\frac{1}{8}\epsilon$-neighbourhood of U_k, so that $\{V_k\}$ is a uniform countable $\frac{1}{2}\epsilon$-covering of C. Take n sufficiently large; the sets

$$W(k, d_1, \ldots, d_n) = \{(x, y) \mid x \in V_k, y \in B, y_i = d_i \text{ for } i \leqslant n\}$$

(where $k = 1, 2, \ldots$, and $d_i \in D_i$) form a uniform star-countable ϵ-covering of $C \times B$. Thus every subspace of $C \times B$ is also an M_c-space with a dense subset of $\leqslant \mathfrak{m}$ points.

For the converse, we need two lemmas. The first is due to S. Ginsburg and J. R. Isbell; see (**3**) for the proof.

LEMMA 1. *If $\mathscr{V}$ is a countable covering (of any space), and the covering $\mathscr{U}$ is a star-star refinement of $\mathscr{V}$ in the sense of* (**10**), *then there is a countable covering $\mathscr{W}$, refined by $\mathscr{U}$ and star-refining $\mathscr{V}$.*

LEMMA 2. *A connected linear graph G in which each vertex belongs to at most $\aleph_0$ edges has at most $\aleph_0$ vertices.*

For, if v is a fixed vertex of G, induction shows that at most $\aleph_0$ vertices of G can be joined to v by a path of not more than n steps $(n = 1, 2, \ldots)$.

Now suppose that (R, ρ) is an M_c-space, with a dense subset of $\leqslant \mathfrak{m}$

points. We construct recursively a sequence of uniform star-countable coverings $\mathscr{U}_n$ $(n = 1, 2, \ldots)$ of R, with the properties: (i) $\mathscr{U}_n$ is refined by the covering by all ϵ_n-spheres $\{S(x, \epsilon_n)\}$ for some $\epsilon_n > 0$, where we may suppose $\epsilon_1 > \epsilon_2 > \ldots$, (ii) each $U \in \mathscr{U}_n$ has diameter $< \min(n^{-1}, \frac{1}{3}\epsilon_{n-1})$ if $n > 1$. To start the induction, $\mathscr{U}_1$ is any uniform star-countable covering by sets of diameter less than 1. From (ii), the covering $\mathscr{U}_k^*$ (consisting of the stars $\mathrm{St}(U, \mathscr{U}_k)$ where $U \in \mathscr{U}_k$) is a star-refinement of $\mathscr{U}_{k-1}$. Consider the components $N_{n\alpha}$ of the nerve N_n of $\mathscr{U}_n$: applying Lemma 2 to their 1-skeletons we see that the sets of $\mathscr{U}_n$ which correspond to vertices in $N_{n\alpha}$ form a countable subcollection $\mathscr{U}_{n\alpha}$ of $\mathscr{U}_n$. Write

$$R_{n\alpha} = \bigcup \{U \mid U \in \mathscr{U}_{n\alpha}\};$$

thus, for fixed n, the sets $R_{n\alpha}$ are open and cover R, and every two have distance not less than ϵ_n.

Fixing n and α for the present, we write

$$\mathscr{V}(n, \alpha, 0) = \mathscr{U}_{n\alpha},$$

a countable covering of $R_{n\alpha}$, and apply Lemma 1 to $\mathscr{V}(n, \alpha, 0)$ and the restriction $\mathscr{U}_{n+1} \mid R_{n\alpha}$ (consisting of those sets of $\mathscr{U}_{n+1}$ which meet, and so are contained in, $R_{n\alpha}$). We obtain a countable covering $\mathscr{V}(n, \alpha, 1)$ of $R_{n\alpha}$ which star-refines $\mathscr{V}(n, \alpha, 0)$ and is refined by $\mathscr{U}_{n+1} \mid R_{n\alpha}$. Repeating the argument, we obtain countable coverings $\mathscr{V}(n, \alpha, k)$ $(k = 1, 2, \ldots)$ of $R_{n\alpha}$ such that $\mathscr{V}(n, \alpha, k)$ star-refines $\mathscr{V}(n, \alpha, k-1)$ and is refined by $\mathscr{U}_{n+k} \mid R_n$. The normal sequence $\mathscr{V}(n, \alpha, k)$ $(k = 0, 1, \ldots)$ determines a pseudometric $\rho_{n\alpha}$ on $R_{n\alpha}$ [see (**10**)] such that, for x, y in $R_{n\alpha}$ and $k = 0, 1, \ldots$, we have

$$\rho_{n\alpha}(x, y) < 2^{-k-2} \Rightarrow x, y \text{ are in the same set of } \mathscr{V}(n, \alpha, k)$$
$$\Rightarrow \rho_{n\alpha}(x, y) \leqslant 2^{-k-1}. \quad (2)$$

By identifying points of $R_{n\alpha}$ at zero $\rho_{n\alpha}$-distance, we obtain a metric space $(R_{n\alpha}^*, \rho_{n\alpha})$ which is separable (for, if we choose one point in each non-empty set of each $\mathscr{V}(n, \alpha, k)$, we get a countable dense set). Hence [(**2**) 187] $(R_{n\alpha}^*, \rho_{n\alpha})$ is imbeddable isometrically in the space C. Thus, by identification and isometry, we obtain a continuous mapping

$$f_{n\alpha} : R_{n\alpha} \to C.$$

Now let α and n vary, and, for each x in R and $n = 1, 2, \ldots$, let $\alpha_n(x)$ be the α for which $x \in R_{n\alpha}$. We take for each n a copy C_n of C, and let D_n denote the set of indices α, which clearly has cardinal $\leqslant \mathfrak{m}$. Let

$$K = \prod C_n, \qquad B = \prod D_n$$

(with the metric introduced in § 2); we define a mapping f of R in $K \times B$ by the rule:

$$f(x) = (k(x), b(x)),$$

where

$$k(x) = \{f_{n,\alpha_n(x)}(x)\} \in \prod C_n = K, \qquad b(x) = \{\alpha_n(x)\} \in \prod D_n = B.$$

We verify that f provides a uniform imbedding of R.

Suppose that $x, y \in R$ and $\rho(x, y) < \epsilon_n$ for some n. Then x, y are in the same set of $\mathscr{U}_i$ for each $i \leqslant n$; thus

$$\alpha_i(x) = \alpha_i(y) = \alpha_i \quad \text{say},$$

for $i = 1, 2, \ldots, n$, and so $b(x)$ and $b(y)$ have distance less than n^{-1} in B. Again, x and y are in the same set of $\mathscr{U}_{n+1-i} | R_{i\alpha_i}$, and so in the same set of $\mathscr{V}(i, \alpha_i, n-i)$, for $i \leqslant n$; hence, by (2), $\rho_{i\alpha_i}(x, y) \leqslant 2^{i+1-n}$. The isometric property of $f_{i\alpha_i}$ now shows that $k(x)$, $k(y)$ have distance less than 2^{-n} in K. Thus f is uniformly continuous.

Conversely, suppose that $x, y \in R$ and $\rho(x, y) > n^{-1}$. Then x, y are not in the same set of $\mathscr{U}_n$. Either $\alpha_n(x) \neq \alpha_n(y)$, in which case $b(x)$, $b(y)$ have distance not less than n^{-1}, or

$$\alpha_n(x) = \alpha_n(y) = \alpha, \quad \text{say},$$

but x, y are not in the same set of $\mathscr{U}_{n\alpha} = \mathscr{V}(n, \alpha, 0)$, and (2) gives

$$\rho_{n\alpha}(x, y) \geqslant 2^{-n-2}.$$

Thus $k(x)$, $k(y)$ have distance $\geqslant 2^{-2n-2}$ in K. This shows both that f is (1–1) and that f^{-1} is uniformly continuous.

Finally, the separable metric space K can be isometrically imbedded in C, and we can adjoin superfluous elements to each D_n to make its cardinal exactly $\mathfrak{m}$, thus obtaining the imbedding of (R, ρ) asserted in the theorem.

The analogous theorem for M_f-spaces, '*there is a universal separable M_f-space*', would be false. For, if X were such a space, we could construct an infinite tree R of unit segments, which (roughly speaking) ramifies faster than any of the components of the nerves of a sequence of countable star-finite coverings forming a uniform basis for X, and could deduce that R could not be uniformly imbedded in X; I omit the details. We shall see below (Theorem 4′) that, at least when $\mathfrak{m} \geqslant \mathfrak{c}$, the analogue of Theorem 3 for M_b-spaces is true; it would be interesting to know if it is valid without this restriction, or if there is a similar result for M_f-spaces. First, we derive the following result, which is weaker in general but applies to separable spaces too.

THEOREM 4. *For each infinite cardinal* $\mathfrak{m}$, *there is an* M_f*-space, with a dense set of cardinal* $\mathfrak{m}$, *in which every* M_b*-space with a dense set of cardinal* $\leqslant \mathfrak{m}$ *can be uniformly imbedded.*

The proof requires the following lemma:

LEMMA 3. *For each positive integer* m, *there is a star-finite countable metric complex* K_m, *of which every connected star-bounded complex of bound* m *is a subcomplex.*†

We shall realize K_m in Hilbert space, with its vertices at the unit points. Enumerating these unit points, we let V_0 consist of the first of them, V_1 consist of the next m, V_2 of the next m^2, and so on; V_k will consist of m^k points. We join (by 1-cells) every member of V_k to every other member of V_k and to every member of V_{k-1} ($k = 1, 2, \ldots$), producing a star-finite linear graph G_m. For every finite set of vertices W of G_m, every two of which are joined, we adjoin the cell spanned by W. This produces the complex K_m. If L is any connected complex in which each vertex is joined to at most m others, we pick a vertex l_0 of L and map it onto the vertex in V_0. The vertices of L to which l_0 is joined can be mapped in a (1–1) way in V_1; the vertices of L joined to these and not yet dealt with are mapped in V_2, and so on. This imbeds the 0-skeleton of L in K_m simplicially, and by linear extension we obtain an imbedding (preserving barycentric coordinates) of L in K_m.

Proof of Theorem 4. We replace the metric ρ_m, on the complex K_m of Lemma 3, by $\min(\rho_m, 1)$, and let P denote the discrete union of the spaces K_m ($m = 1, 2, \ldots$), taking the distance between points of distinct sets K_m to be 1. Let Q be the product of countably many copies P_n of P ($n = 1, 2, \ldots$), and let B be the generalized Baire space $\prod D_n$, where D_n is a discrete set of cardinal $\mathfrak{m}$. Then $Q \times B$ has a dense set of cardinal $\mathfrak{m}$ and is an M_f-space because each factor P_n or D_n is an M_f-space. Given any M_b-space (R, ρ), take a basis of uniform star-bounded coverings $\mathscr{U}_n$ ($n = 1, 2, \ldots$) of R; we may suppose $\mathscr{U}_n$ to be a normal sequence. As in the proof of Theorem 3, let $N_{n\alpha}$ be the components of the nerve N_n of $\mathscr{U}_n$: they correspond to a partitioning of R into disjoint open sets $R_{n\alpha}$. From Lemma 3, each $N_{n\alpha}$ can be imbedded isometrically (so far as distances less than 1 are concerned) in P, so that the uniformly continuous canonical mapping of R in N_n [cf. (**4**) 69] gives rise to a uniformly continuous mapping $f_n : R \to P_n$. Write

$$q(x) = \{f_n(x)\} \in Q, \qquad b(x) = \{\alpha_n(x)\} \in B,$$

† By a 'star-bounded complex of bound m' we mean a complex in which each vertex belongs to at most m 1-cells.

where $\alpha_n(x)$ is defined as in the proof of Theorem 3. Let

$$f(x) = (q(x), b(x));$$

then f is the desired uniform imbedding of (R, ρ) in $Q \times B$.

THEOREM 4'. *For each cardinal* $\mathfrak{m} \geqslant \mathfrak{c}$, *there is an* M_b*-space with a dense set of cardinal* $\mathfrak{m}$, *in which every* M_b*-space with a dense set of cardinal* $\leqslant \mathfrak{m}$ *can be uniformly imbedded.*

The proof is similar to that of Theorem 4; we modify the definition of P and Q as follows. Consider the connected star-bounded complexes $K_\alpha(m)$ of fixed bound m (i.e. each vertex is incident with at most m edges); there are at most $\mathfrak{c}$ of them since (from Lemma 2) each has at most $\aleph_0$ vertices. As in Theorem 4 we form the discrete union P_m of all the spaces $K_\alpha(m)$, for each m $(m = 1, 2, \ldots)$, taking the distance between x and y in P_m to be the same as in $K_\alpha(m)$ if they are both in $K_\alpha(m)$ and have distance less than 1 there, and to be 1 otherwise. Let

$$Q = \prod P_m, \qquad B = \prod D_n,$$

where D_n is discrete and of cardinal $\mathfrak{m}$. Then $Q \times B$ is the universal M_b-space required. The only point in the proof which now requires care is that f^{-1} is uniformly continuous, where f is the obvious imbedding map; it is at this point that the similar argument applied to M_f-spaces would break down.

Though the uniform properties of M_b and M_f spaces present difficulties, their topological properties are easily described. We call a topological space R an *M-space* if

[5.1] *R is homeomorphic to a subspace of a metrizable S-space.*†

It has been shown by K. Morita [(**6**) 361; see also (**8**) 169] that [5.1] is equivalent to each of:

[5.2] *R is homeomorphic to a subspace of the product $H \times B$ of the Hilbert cube with some generalized Baire space,*

[5.3] *R is regular and has a basis of open sets consisting of the members of a sequence of star-finite (or star-countable) coverings of R.*‡

Other characterizations of M-spaces are easily deduced: for example, that R is to have a metric for which its completion is an S-space.

† An *S-space* is one in which every covering has a star-finite refinement. A subspace of a metrizable S-space need not be an S-space [(**8**) 169].

‡ Since Morita's proofs have not been published, I remark that the implication [5.3] → [5.2] can be proved by an argument similar to that proving Theorem 4 (the metrizability of R follows from the Nagata–Smirnov theorem); the other implications are straightforward.

THEOREM 5. *A necessary and sufficient condition that a space R admits an M_b, or M_f, or M_c uniformity, is that R is an M-space.*

If R admits an M_c uniformity, Theorem 3 shows that R satisfies [5.2], and so is an M-space. On the other hand, in its usual metric, $H \times B$ is an M_b-space (by an argument similar to the first part of the proof of Theorem 3), so that [5.2] shows that every M-space admits an M_b-uniformity, which automatically makes it also an M_f and M_c space.

COROLLARY. *R is an M-space if and only if it has a metric in which every uniform covering has a star-bounded (or star-finite, or star-countable) refinement.*

For, if R has a metric ρ in which every uniform covering has a star-countable refinement (not necessarily uniform), then by constructing a suitable normal sequence of star-countable coverings of R, we obtain (in a standard way) a metric ρ' (finer than ρ) such that (R, ρ') is an M_c-space. Hence R is an M-space, and admits an M_b-uniformity.

Remark. The equivalence, in this corollary, of the alternatives 'star-finite' and 'star-countable' is obvious directly: for, by combining Lemma 2 with a theorem of Morita, we see that every star-countable covering of a metric (or even countably paracompact normal) space has a star-finite refinement. But in general it need not have a star-bounded refinement.

Similarly, the third possibility 'star-bounded' can be introduced into [5.3] without changing its meaning; but this is obvious directly since (from Lemma 2) we may replace each star-countable covering in [5.3] by a sequence of star-bounded ones without spoiling the basis property.

6. Further problems

It is natural to ask if there are universal spaces for those M_b-spaces of uniform density $\leqslant d$, for fixed d.† In the simplest cases, $d = 0$, 1, an affirmative answer follows easily. I do not know the answer for larger values of d. Similar questions also arise in connexion with dimension.

A different type of space for which the existence of universal spaces would be of interest is that of the metrizable uniformities with a basis of *locally finite* uniform coverings. It is easy to see (by constructing a suitable normal sequence of locally finite coverings) that every

† Here a uniform space is said to have 'uniform density $\leqslant d$' if it has a basis of uniform coverings $\mathscr{U}_\lambda$ such that each U in $\mathscr{U}_\lambda$ meets at most d other sets of $\mathscr{U}_\lambda$. Clearly, if R has uniform density $\leqslant 1$, it has uniform density $\leqslant 0$.

metrizable space (R, ρ) has a metric ρ' such that (R, ρ') has a basis of locally finite (uniform) coverings. The particular interest of this type of space is that it seems possible (though unlikely) that every metric on R might be a suitable ρ'. In fact, it seems to be unknown whether or not, for an arbitrary uniform space, every uniform covering has a *uniform* locally finite refinement.

REFERENCES

1. M. Atsuji, 'Uniform continuity of continuous functions of metric spaces', *Pacific J. of Math.* 8 (1958) 11–16.
2. S. Banach, *Théorie des opérations linéaires* (Warsaw 1932).
3. S. Ginsburg and J. R. Isbell, 'Some operators on uniform spaces', *Trans. American Math. Soc.* 93 (1959) 145–68.
4. J. R. Isbell, 'Euclidean and weak uniformities', *Pacific J. of Math.* 8 (1958) 67–86.
5. C. Kuratowski, *Topologie* I (Warsaw 1948).
6. K. Morita, 'Normal families and dimension theory for metric spaces', *Math. Annalen* 128 (1954) 350–62.
7. J. Nagata, 'On the uniform topology of bicompactifications', *J. Inst. Pol. Osaka City Univ.* 1 (1950) 28–38.
8. —— 'Note on dimension theory for metric spaces', *Fund. Math.* 45 (1958) 143–81.
9. P. Samuel, 'Ultrafilters and compactifications of uniform spaces', *Trans. American Math. Soc.* 64 (1948) 100–32.
10. J. W. Tukey, *Convergence and uniformity in topology* (Princeton 1940).

THE NON-SYMMETRIC PUNCH PROBLEM IN LAYERED MEDIA BY THE WIENER–HOPF METHOD

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Summary

THE deformation in an elastic layer resting upon a smooth, rigid base due to non-symmetric punch has been investigated by the method of Wiener-Hopf for the solution of non-homogeneous integral equations. The stresses under the punch as well as the deformation on the free surface have been computed for a prescribed form of the punch.

1. Introduction

The determination of stresses and deformation in layered media has been the subject of investigation by Burmister (**2**, **3**), Acum and Fox (**1**), Paria (**9**), and others for its application in soil mechanics, foundation and highway engineering, and geophysics. Usually either the stresses or the deformations are prescribed on the bounding surfaces; a mixed condition of both may be prescribed leading to the punch or crack problems. When the forces and deformations are axially symmetric, the punch problem reduces to the solution of dual integral equations [Sneddon (**12**)]. The situation is more complex for non-symmetric boundary conditions. The theory of sectionally holomorphic functions and Hilbert's linear relationship can then be applied. Though this method is quite general, its application is not easy unless the boundaries are such that they can be transformed to a half-plane or a unit circle by simple mapping functions [Muskhelishvili (**8**), Paria (**10**)]. Wiener–Hopf technique with its own difficulties is another powerful method specially suited for those boundaries where integral transforms such as those of Fourier, Mellin, and Hankel, can be applied. This technique has been used quite successfully in electromagnetic theory, diffraction problems, acoustic waves, and water waves. Extensive references to these can be found in a paper by Heins (**5**). Carrier and Munk (**4**) have treated a problem of diffusion of tides by this method. Applications to elastic problems are, however, few. Only the work of Koiter (**6**, **7**) and Paria (**11**) need be mentioned in this connexion. In

the present paper deformation in an elastic layer resting on a smooth rigid base has been considered by the Wiener–Hopf method when a non-symmetric punch is applied on a part of the boundary, the remaining part being kept free. Forces required to hold the punch as well as resulting deformation on the free part of the boundary have been determined.

2. Statement of the problem

We consider the plain strain distribution in a layer of elastic material resting on a smooth rigid base. The bounding surfaces of the layer are given by $y = 0$ and $y = b$, the positive direction of the axis of y being taken vertically downwards. The vertical component of the displacement is prescribed on $y = 0$, $0 < x < \infty$ by pressing a smooth rigid block against the medium so that no shearing stresses are produced under it, though the horizontal components of displacement are not prevented thereby. The part $y = 0$, $-\infty < x < 0$ is assumed to be stress-free. On the lower surface of the layer both the vertical component of displacement as well as the shearing forces vanish. It is therefore required to find the solution of the equation

$$\nabla_1^4 \chi = 0, \quad \text{where} \quad \nabla_1^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2, \tag{1}$$

for the stress function χ with the boundary conditions

$$\tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} = 0 \quad \text{on } y = 0 \text{ and } y = b \quad (-\infty < x < \infty), \tag{2}$$

$$\sigma_y = \frac{\partial^2 \chi}{\partial x^2} = \begin{cases} -m_+(x) & \text{on } y = 0 \quad (0 < x < \infty), \\ 0 & \text{on } y = 0 \quad (-\infty < x < 0), \end{cases} \tag{3}$$

$$v = 0 \quad \text{on } y = b \quad (-\infty < x < \infty), \tag{4}$$

where $m_+(x)$ is an *unknown* function to be determined from the condition that the y-component of displacement $v(x, y)$ has the value

$$v(x, 0) = \phi_+(x) \quad \text{on } y = 0 \quad (0 < x < \infty), \tag{5}$$

the function $\phi_+(x)$ being *prescribed*. Also the function χ with all its derivatives must vanish at a great distance. Let

$$v(x, 0) = \phi_-(x) \quad \text{on } y = 0 \quad (-\infty < x < 0). \tag{6}$$

The function $\phi_-(x)$ is to be determined as a part of the solution. We may write

$$v(x, 0) \equiv \phi(x) = \phi_+(x) + \phi_-(x), \tag{7}$$

where the first or the second member on the right-hand side is zero according as x is negative or positive.

3. Reduction to Wiener–Hopf equation

We introduce the complex Fourier transform $\bar{\chi}$ of χ defined by Titchmarsh (13) as

$$\bar{\chi}(\zeta, y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \chi(x, y)\exp(i\zeta x)\, dx \quad (\zeta = \xi + i\eta), \tag{8}$$

which is regular in the strip $-a_1 < \eta < a_1$ $(a_1 > 0)$ if it is assumed that $\chi\exp(-\lambda x)$ belongs to $L(-\infty, \infty)$ for all values of λ in the range $-a_1 < \lambda < a_1$. If it is further assumed that

$$\lim_{|x|\to\infty} \frac{\partial^r \chi}{\partial x^r}\exp(-\lambda x) = 0 \quad (r = 1, 2, 3), \tag{9}$$

the relations from (1) to (4) transform respectively to

$$\frac{\partial^4 \bar{\chi}}{\partial y^4} - 2\zeta^2 \frac{\partial^2 \bar{\chi}}{\partial y^2} + \zeta^4 \bar{\chi} = 0, \tag{10}$$

$$\frac{\partial \bar{\chi}}{\partial y} = 0 \qquad \text{on } y = 0 \text{ and } y = b, \tag{11}$$

$$\zeta^2 \bar{\chi} = M_+(\zeta) \quad \text{on } y = 0, \tag{12}$$

$$\bar{v}(\zeta, y) = 0 \qquad \text{on } y = b, \tag{13}$$

where

$$M_+(\zeta) = (2\pi)^{-\frac{1}{2}} \int_0^{\infty} m_+(x)\exp(i\zeta x)\, dx \tag{14}$$

is regular for $\eta > a_2 > 0$ $(a_2 < a_1)$ if $m_+(x)\exp(-\lambda x)$ belongs to $L(0, \infty)$ for all $\lambda > a_2$. Introducing the generalized transforms

$$\bar{\phi}_+(\zeta) = (2\pi)^{-\frac{1}{2}} \int_0^{\infty} \phi_+(x)\exp(i\zeta x)\, dx, \tag{15}$$

$$\bar{\phi}_-(\zeta) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} \phi_-(x)\exp(i\zeta x)\, dx, \tag{16}$$

which are assumed to be regular in the half-planes $\eta > a_2$ and $\eta < -a_2$ respectively, we write (7) in the form

$$v(x, 0) \equiv \phi(x) = (2\pi)^{-\frac{1}{2}} \int_{ia-\infty}^{ia+\infty} \bar{\phi}_+(\zeta)\exp(-i\zeta x)\, d\zeta + (2\pi)^{-\frac{1}{2}} \int_{-ia-\infty}^{-ia+\infty} \bar{\phi}_-(\zeta)\exp(-i\zeta x)\, d\zeta, \tag{17}$$

where $a_2 < a < a_1$. The displacement component $v(x, y)$ can be expressed in terms of the transformed stress function $\bar{\chi}$ [Sneddon (**12**)] as

$$v(x,y) = \{(1+\sigma)/E\}(2\pi)^{-\frac{1}{2}} \int_{ia-\infty}^{ia+\infty} \left\{(1-\sigma)\zeta^{-2}\frac{\partial^3\bar{\chi}}{\partial y^3} - (2-\sigma)\frac{\partial\bar{\chi}}{\partial y}\right\}\exp(-i\zeta x)\,d\zeta, \tag{18}$$

where σ is the Poisson ratio and E is the Young modulus. To satisfy the equation (10) we take

$$\bar{\chi}(\zeta, y) = (A+By)\cosh(\zeta y)+(C+Dy)\sinh(\zeta y), \tag{19}$$

where A, B, C, D are independent of y. The conditions from (11) to (13) are satisfied if

$$\zeta^2 A = M_+(\zeta), \qquad B+\zeta C = 0,$$

$$(C/A) = -\{2\sinh^2(b\zeta)\}/\{2b\zeta+\sinh(2b\zeta)\}, \tag{20}$$

$$(D/A)\{b\zeta\cosh(b\zeta)+\sinh(b\zeta)\} = \zeta\sinh(b\zeta)\{(b\zeta C/A)-1\}. \tag{21}$$

Substituting in (18) the value of $\bar{\chi}$ given by (19) and (20), putting $y = 0$, and then equating with (17), we obtain

$$\int_{ia-\infty}^{ia+\infty} \{\sigma_1 bM_+(\zeta)H(\zeta)-\bar{\phi}_+(\zeta)\}\exp(-i\zeta x)\,d\zeta - \int_{-ia-\infty}^{-ia+\infty} \bar{\phi}_-(\zeta)\exp(-i\zeta x)\,d\zeta = 0 \tag{22}$$

with

$$\sigma_1 = 4(1-\sigma^2)/E, \tag{23}$$

$$H(\zeta) = \{\cosh(2b\zeta)-1\}/\{2b\zeta+\sinh(2b\zeta)\}. \tag{24}$$

If $H(\zeta)$ is regular at least in the strip $a_2 < \eta < a_1$, the integrand of the first integral on the left-hand side of (22) is regular in this strip, and by a general theorem stated by Titchmarsh (**13**) the function $\bar{\phi}_-(\zeta)$ is regular in the extended half-plane $\eta < a_1$, and in this region the equation

$$\sigma_1 bM_+(\zeta)H(\zeta)-\bar{\phi}_+(\zeta) = \bar{\phi}_-(\zeta) \tag{25}$$

holds.

The Wiener–Hopf equation (25) is to be solved from the conditions already imposed, viz.

(i) $M_+(\zeta)$ is regular in the half-plane $\eta > a_2$,

(ii) $\bar{\phi}_-(\zeta)$ is regular in the half-plane $\eta < a_1$,

(iii) both $M_+(\zeta)$ and $\bar{\phi}_-(\zeta)$ tend to zero as $|\zeta| \to \infty$ in their respective half-planes of regularity.

It is seen from (24) that $H(\zeta)$ tends to $\frac{1}{4}$ as $\zeta \to 0$ and $H(\zeta) \sim (2b\zeta)^{-1}$ as $|\zeta| \to \infty$. We write

$$H(\zeta) = (4b^2\zeta^2+16)^{-\frac{1}{2}}G(\zeta), \tag{26}$$

where

$$G(\zeta) = (4b^2\zeta^2+16)^{\frac{1}{2}}(\cosh 2b\zeta-1)/\{2b\zeta+\sinh(2b\zeta)\} \tag{27}$$

is regular and non-zero in the strip $a_2 < \eta < a_1$ ($a_2 < 2b^{-1}$), and in this

strip $G(\zeta) \to 1$ when $\zeta \to 0$ or $|\zeta| \to \infty$. By Cauchy's theorem $G(\zeta)$ can be written [Titchmarsh (**13**)] as

$$G(\zeta) = G_+(\zeta)/G_-(\zeta), \tag{28}$$

with

$$\log G_+(\zeta) = \frac{1}{2\pi i} \int_{ia_2-\infty}^{ia_2+\infty} [\{\log G(z)\}/(z-\zeta)]\, dz, \tag{29}$$

$$\log G_-(\zeta) = \frac{1}{2\pi i} \int_{ia_1-\infty}^{ia_1+\infty} [\{\log G(z)\}/(z-\zeta)]\, dz, \tag{30}$$

where $G_+(\zeta)$ and $G_-(\zeta)$ are regular and free from zeros in the half-planes $\eta > a_2$ and $\eta < a_1$ respectively. Both of them tend to unity as $|\zeta| \to \infty$ in their half-planes of regularity. The equation (25) takes the form

$$\sigma_1 b M_+(\zeta) G_+(\zeta)(2b\zeta+4i)^{-\frac{1}{2}} - \psi(\zeta) = (2b\zeta-4i)^{\frac{1}{2}} \bar{\phi}_-(\zeta) G_-(\zeta), \tag{31}$$

where

$$\psi(\zeta) = (2b\zeta-4i)^{\frac{1}{2}} \bar{\phi}_+(\zeta) G_-(\zeta). \tag{32}$$

Again, if $\psi(\zeta)$ is regular and tends to zero as $|\zeta| \to \infty$ in the strip $a_2 < \eta < a_1$, it can be written as

$$\psi(\zeta) = \psi_+(\zeta) - \psi_-(\zeta) \tag{33}$$

with

$$\psi_+(\zeta) = \frac{1}{2\pi i} \int_{ia_2-\infty}^{ia_2+\infty} \{\psi(z)/(z-\zeta)\}\, dz, \tag{34}$$

$$\psi_-(\zeta) = \frac{1}{2\pi i} \int_{ia_1-\infty}^{ia_1+\infty} \{\psi(z)/(z-\zeta)\}\, dz, \tag{35}$$

where $\psi_+(\zeta)$ and $\psi_-(\zeta)$ have the same properties as $G_+(\zeta)$ and $G_-(\zeta)$ respectively. Equation (31) reduces to

$$\sigma_1 b M_+(\zeta) G_+(\zeta)(2b\zeta+4i)^{-\frac{1}{2}} - \psi_+(\zeta) = (2b\zeta-4i)^{\frac{1}{2}} \bar{\phi}_-(\zeta) G_-(\zeta) - \psi_-(\zeta) = P(\zeta). \tag{36}$$

In the above equalities the first member is regular in the half-plane $\eta > a_1$, the second member is regular in the overlapping half-plane $\eta < a_2$ ($a_2 < 2b^{-1}$), and hence each is the analytical continuation of the other, so that $P(z)$ is an entire function. Thus

$$\sigma_1 b M_+(\zeta) = (2b\zeta+4i)^{\frac{1}{2}} \{\psi_+(\zeta) + P(\zeta)\}/G_+(\zeta), \tag{37}$$

$$\bar{\phi}_-(\zeta) = (2b\zeta-4i)^{-\frac{1}{2}} \{\psi_-(\zeta) + P(\zeta)\}/G_-(\zeta). \tag{38}$$

The inverse transforms of (37) and (38) are

$$\sigma_1 bm_+(x) = (2\pi)^{-\frac{1}{2}} \int_{ia-\infty}^{ia+\infty} \{(2b\zeta+4i)^{\frac{1}{2}}/G_+(\zeta)\}\{\psi_+(\zeta)+P(\zeta)\}\exp(-i\zeta x)\,d\zeta, \tag{39}$$

$$\phi_-(x) = (2\pi)^{-\frac{1}{2}} \int_{ia-\infty}^{ia+\infty} \{(2b\zeta-4i)^{-\frac{1}{2}}/G_-(\zeta)\}\{\psi_-(\zeta)+P(\zeta)\}\exp(-i\zeta x)\,d\zeta. \tag{40}$$

4. Prescribed surface displacement

If $\phi_+(x) = \kappa_1 \exp(-\kappa_2 x)$, we have

$$\bar{\phi}_+(\zeta) = (2\pi)^{-\frac{1}{2}}\{i\kappa_1/(\zeta+i\kappa_2)\}. \tag{41}$$

From (32), (34), (41),

$$\psi_+(\zeta) = \frac{i\kappa_1}{(2\pi)^{\frac{1}{2}}}(2\pi i)^{-1} \int_{ia_1-\infty}^{ia_2+\infty} [\{(2bz-4i)^{\frac{1}{2}}G_-(z)\}/\{(z+i\kappa_2)(z-\zeta)\}]\,dz.$$

Considering the region $\operatorname{im} z < a_2$ and calculating the residue at the pole $z = -i\kappa_2$, we obtain

$$\psi_+(\zeta) = \{(i\kappa_1)/(2\pi)^{\frac{1}{2}}\}(-i)^{\frac{1}{2}}(2b\kappa_2+4)^{\frac{1}{2}}G_-(-i\kappa_2)/(\zeta+i\kappa_2). \tag{42}$$

When (42) is used, the first member of (36) tends to zero as $|\zeta| \to \infty$, and hence $P(z) \equiv 0$. With the help of (32), (41), (42) the relations (39) and (40) simplify to

$$\sigma_1 bm_+(x) = i\kappa_1(-i)^{\frac{1}{2}}(2b\kappa_2+4)^{\frac{1}{2}}G_-(-i\kappa_2)\frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty} \frac{(2b\zeta+4i)^{\frac{1}{2}}}{(\zeta+i\kappa_2)G_+(\zeta)}\exp(-i\zeta x)\,d\zeta, \tag{43}$$

$$\phi_-(x) = \frac{i\kappa_1}{2\pi}\int_{ia-\infty}^{ia+\infty} \left\{\frac{(-i)^{\frac{1}{2}}(2b\kappa_2+4)^{\frac{1}{2}}G_-(-i\kappa_2)}{(2b\zeta-4i)^{\frac{1}{2}}G_-(\zeta)} - 1\right\}\frac{\exp(-i\zeta x)}{\zeta+i\kappa_2}\,d\zeta. \tag{44}$$

If Q be the total surface load, then

$$Q = \int_0^\infty m_+(x)\,dx.$$

From (43),

$$\sigma_1 bQ = \kappa_1(-i)^{\frac{1}{2}}(2b\kappa_2+4)^{\frac{1}{2}}G_-(-i\kappa_2)\frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty} \frac{(2b\zeta+4i)^{\frac{1}{2}}}{\zeta(\zeta+i\kappa_2)G_+(\zeta)}\,d\zeta. \tag{45}$$

5. Approximate solution

As a first approximation we take

$$G(\zeta) = 1. \tag{46}$$

It is seen from Table I that for real values of ζ the assumption (46)

involves a maximum error of 19 per cent. corresponding to $2b\zeta = 4{\cdot}5$. More terms can be taken on the right-hand side of (46) to minimize this error to any desired degree. However, the procedure will be the same even when the corrections are introduced [Koiter (**6**)]. From (29) and (30),

$$G_+(\zeta) = G_-(\zeta) = 1. \tag{47}$$

The relations (43) and (44) reduce to

$$\sigma_1 b m_+(x) = i\kappa_1(-i)^{\frac{1}{2}}(2b\kappa_2+4)^{\frac{1}{2}}\frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty}\{(2b\zeta+4i)^{\frac{1}{2}}/(\zeta+i\kappa_2)\}\exp(-i\zeta x)\,d\zeta, \tag{48}$$

$$\phi_-(x) = \frac{i\kappa_1}{2\pi}\int_{ia-\infty}^{ia+\infty}\left\{\frac{(-i)^{\frac{1}{2}}(2b\kappa_2+4)^{\frac{1}{2}}}{(2b\zeta-4i)^{\frac{1}{2}}}-1\right\}\frac{\exp(-i\zeta x)}{\zeta+i\kappa_2}\,d\zeta. \tag{49}$$

The above integrals are evaluated by considering the respective regions $\eta < a$ and $\eta > a$. In the first $\zeta = -i\kappa_2$ is the pole and $\zeta = -2(i/b)$ is the branch point, while for the second $\zeta = 2i/b$ is the branch point. Remembering that

$$i^{\frac{1}{2}} = \exp(\tfrac{1}{4}i\pi), \qquad (-i)^{\frac{1}{2}} = \exp(-\tfrac{1}{4}i\pi),$$

we obtain finally

$$\sigma_1 b m_-(x) = \kappa_1(16-4b^2\kappa_2^2)^{\frac{1}{2}}\exp(-\kappa_2 x)+ \\ +2(\kappa_1/\pi)(4+2b\kappa_2)^{\frac{1}{2}}\exp(-2x/b)I_1(x) \quad (\kappa_2 < 2/b), \tag{50}$$

$$\phi_-(x) = 2(\kappa_1/\pi)(4+2b\kappa_2)^{\frac{1}{2}}\exp(2x/b)I_2(x), \tag{51}$$

where

$$I_1(x) = \int_0^\infty t^2(t^2+4-2b\kappa_2)^{-1}\exp\{-xt^2/(2b)\}\,dt \quad (x > 0), \tag{52}$$

$$I_2(x) = \int_0^\infty (t^2+4+2b\kappa_2)^{-1}\exp\{xt^2/(2b)\}\,dt \qquad (x < 0). \tag{53}$$

The integrals (52) and (53) have been evaluated by Koiter (**6**) as

$$I_1(x) = \tfrac{1}{2}(2b\pi/x)^{\frac{1}{2}}-\tfrac{1}{2}\pi(4-2b\kappa_2)^{\frac{1}{2}}\exp\{(2b^{-1}-\kappa_2)x\}\operatorname{erfc}\{(2b^{-1}-\kappa_2)x\}^{\frac{1}{2}} \\ (x > 0), \tag{52 a}$$

$$I_2(x) = \tfrac{1}{2}\pi(4+2b\kappa_2)^{-\frac{1}{2}}\exp\{-(2b^{-1}+\kappa_2)x\}\operatorname{erfc}\{-(2b^{-1}+\kappa_2)x\}^{\frac{1}{2}} \quad (x < 0), \tag{53 a}$$

where

$$\operatorname{erfc}(s) = 1-\frac{2}{\sqrt{\pi}}\int_0^s e^{-t^2}\,dt. \tag{54}$$

The total load Q can be calculated from (45). It is easily found from (50) and (52) as

$$\sigma_1 bQ = 2(\kappa_1/\kappa_2)(4+2b\kappa_2)^{\frac{1}{2}}. \tag{55}$$

6. Numerical results

For simplicity we take $\kappa_2 = 1/b$. Then the relations (50) and (51) with the help of (52 a), (53 a), (55) reduce respectively to

$$\sqrt{2}(b/Q)m_+(x) = \exp(-x/b)+\sqrt{2}\pi^{-1}\exp(-2x/b)I_1(x) \quad (x > 0), \quad (56)$$

$$\{2\sqrt{6}/(\sigma_1 Q)\}\phi_-(x) = (2\sqrt{6}/\pi)\exp(2x/b)I_2(x) \quad (x < 0), \quad (57)$$

where $I_1(x)$ and $I_2(x)$ are now given by

$$I_1(x) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}(b/x)^{\frac{1}{2}}-(\pi/\sqrt{2})\exp(x/b)\operatorname{erfc}(x/b)^{\frac{1}{2}} \quad (x > 0), \quad (58)$$

$$I_2(x) = (\pi/2\sqrt{6})\exp(-3x/b)\operatorname{erfc}(-3x/b)^{\frac{1}{2}} \quad (x < 0), \quad (59)$$

Table II gives the values of $\sqrt{2}(b/Q)m_+(x)$ and $\{2\sqrt{6}/(\sigma_1 Q)\}\phi_-(x)$ as functions of x/b. Lastly it may be mentioned that all the assumptions regarding the regularity and behaviour of different functions in their respective domains can be shown to hold.

TABLE I

$2b\zeta$	0·0	1·0	2·0	3·0	4·0	4·5	5·0	6·0
$H(\zeta)$	0·2500	0·2497	0·2454	0·2322	0·2102	0·1976	0·1849	0·162
$(16+4b^2\zeta^2)^{\frac{1}{2}}$	0·2500	0·2425	0·2236	0·2000	0·1768	0·1661	0·1562	0·139
$G(\zeta)$	1·0000	1·0297	1·0975	1·1610	1·1889	1·1896	1·1837	1·166
error %	0·0	3·0	10·0	16·0	19·0	19·0	18·0	17·0

TABLE II

x/b	0·00	0·06	0·25	1·00	2·25	4·00	6·25	9·00
$\sqrt{2}(b/Q)m_+(x)$	$[b/(\pi x)]^{\frac{1}{2}}$	2·5654	1·4681	0·5159	0·0806	0·0183	0·0025	0·0000
$-x/b$	0·00	0·02	0·08	0·33	0·75	1·33	2·08	3·00
$\{2\sqrt{6}/(\sigma_1 Q)\}\phi_-(x)$	1·0000	0·7383	0·5194	0·2188	0·0718	0·0178	0·0033	0·0000

REFERENCES

1. W. E. Acum and L. Fox, *Geotechnique* (London) 11 (1951) 4, 293.
2. D. M. Burmister, *Proc. 23rd Annual Meeting Highways Res. Board, Nov.* 23 (1943) 126.
3. —— *J. Appl. Phys.* 16 (1945) 296.
4. G. F. Carrier and W. H. Munk, *Proc. Symp. Appl. Math.* 5 (1954) 89.
5. A. E. Heins, *Comm. Pure Appl. Math.* 9 (1956) 447.
6. W. T. Koiter, *Proc. Kon. Ned. Akad. Weten. Amsterdam* B, 57 (1954) 558, 565.
7. —— ibid. 59 (1956) 354, 365.
8. N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity* (Holland, 1953).
9. G. Paria, *Bull. Calcutta Math. Soc.* 48 (1956) 75.
10. —— *J. Appl. Mech.* 24 (1957) 1.
11. —— *Bull. Calcutta Math. Soc.* 49 (1957) 37.
12. I. N. Sneddon, *Fourier transforms* (New York, 1951) 404, 431, 462.
13. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Oxford, 1937) 4, 255, 339.

SPACES OF FINITE CHARACTERISTIC

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Introduction

IT is well known that, if the singular homology groups of a simply-connected space are torsion groups in dimensions greater than 1, then so are the homotopy groups. In fact, the orders of the elements of the qth homotopy group will be bounded if the orders of the homology classes in dimensions not exceeding q are bounded: various conjectures have been made about the relation between the bounds. In this paper a crude upper limit for the bound of the orders of homotopy classes will be found, with more reasonable results for suspensions. It suffices to consider only CW complexes since any space may be replaced by its singular complex (**12**) for these purposes.

Particular interest attaches to spaces with one non-trivial homology group of positive dimension (Moore spaces) since these are the bricks with which a complex of given homotopy type can be constructed from the homology groups, and certain invariants (akin to the Postnikov invariants). These can be represented as suspensions in such a way that the homotopy class of the identity map has finite order under track addition (**1**).

A complex A will be said to have *characteristic* $p\ (\leqslant \infty)$ if the homotopy class of the identity map of its suspension has order p under track addition. The *homology characteristic* of a space will mean the least common multiple of the orders of the (singular) homology classes of positive dimension. Thus the real projective plane has homology characteristic 2 and characteristic 4 [(**1**) Part II]. A connected complex of finite (homological) dimension will be shown to have finite characteristic if and only if it has finite homology characteristic; the latter divides the former, and the former a power of the latter.

A simply-connected complex of finite homology characteristic has the homotopy type of a wedge, or, equally, a direct product, of spaces of mutually prime prime-power homology characteristics; in examining the qth homotopy group it suffices to consider a complex with the same homology groups in dimension up to $q+1$, and no others. Therefore it is sufficient to consider complexes of prime-power characteristic.

Quart. J. Math. Oxford (2), 11 (1960), 124–36.

Let p be a prime, and suppose that A is of characteristic p^m, and $(n-1)$-connected ($n \geqslant 1$); let E denote suspension. We have the proposition:

PROPOSITION. *$p^{mk}\pi_q(EA) = 0$ if $q \leqslant 2^k n$. If A is itself a suspension, then*

$$p^{m+k}\pi_q(EA) = 0 \text{ if } q \leqslant p^{k+1}n.$$

The last result may be best-possible: $\pi_{10}(EA)$ is cyclic of order 8 when A is the 3-fold suspension of the real projective plane. Various crude results can be deduced: for example, if $p^r H_t(A) = 0$ for $1 \leqslant t \leqslant q$, then a space B can be found of characteristic at most $p^{r(q+1)}$, with $\pi_q(EA)$ as a homomorphic image of $\pi_q(EB)$. Again, if A is at least simply connected in the proposition, the first result yields a bound of the order of $p^{2^{mk}}$ for $\pi_q(A)$ if $q \leqslant 2^k(n-1)$; for $q \leqslant 2(n-1)$, A can be replaced by a suspension, and a sharper result can be obtained.

Though spaces of any odd characteristic can be constructed, no space of characteristic 2 is known at present, and it is reasonable to conjecture that none exist. An existence problem of a different kind is this: is there a compactum of finite (covering) dimension and finite homology characteristic which has infinitely many non-zero singular homology groups? Such spaces are known (3) with infinite homology characteristic.

1. Characteristic and homology characteristic

The first propositions relate the two types of characteristic; here 'spaces' mean CW complexes with base points, E denotes suspension, and $\pi_1^A(x)$ the track group of base-point preserving homotopy classes of maps $EA \to X$ (as defined in the Appendix). The homotopy class of the identity map $EA \to EA$ is written $\mathbf{1}_A$. We have the lemmas:

LEMMA 1.1. *If $\mathbf{1}_A$ has order $p < \infty$, then A is connected, and $pH_q(A) = 0$ (all $q > 0$).*

For, if $\alpha, \beta\colon EA \to X$, the track-sum $\alpha+\beta\colon EA \to X$ is defined by composing a map $\rho\colon EA \to EA \vee EA$ with a map $(\alpha \vee \beta)\colon EA \vee EA \to X$ (see Appendix); ρ pinches a middle section $A \subset EA$ to a point. It is easily verified that

$$(\alpha+\beta)_* = \alpha_* + \beta_*\colon H_q(EA) \to H_q(X), \quad \text{for } q > 0.$$

Hence, if $z \in H_q(EA)$,

$$pz = p(\mathbf{1}_{A*}z) = (p\mathbf{1}_A)_* z = 0.$$

Also, $$H_q(EA) \approx H_{q-1}(A) \quad (q > 1),$$

and $H_0(A) \approx H_1(EA) + Z$ must be free abelian.

LEMMA 1.2. *If A is connected, and has homology characteristic p, the characteristic of A divides the product of the maximum of the orders of the elements of the cohomology groups $H^q(A; Z_p)$ $(q = 1, 2, \ldots)$.*

The lemma has content only when all but a finite number of the cohomology groups are 0. It is shown in (**5**) and [(**1**) Part II] that there is a sequence of groups

$$0 = Q_0, Q_1, \ldots, Q_r, \ldots, Q_\infty = \pi_1^A(X),$$

such that Q_r is a central extension of a subgroup of Q_{r-1} by a factor group of $H^r(A; \pi_{r-1}(X))$ for each $r \geqslant 1$; Q_∞ is the inverse limit of the system of groups Q_r and projections $Q_r \to Q_{r-1}$. The universal-coefficient theorem and the conditions imply that the maximum order of elements of $H^r(A; \pi_{r-1}(X))$ divides that of $H^r(A; Z_p)$.† The lemma follows by induction on r.

For simply-connected complexes of finite dimension, (1.1) and (1.2) imply that finite characteristic and finite homology characteristic are equivalent properties.

LEMMA 1.3. *If A is 1-connected and has homology characteristic p, then for any $q > 1$, $p^s \pi_q(A) = 0$ for some power s (depending on q).*

This follows from Serre's C-theory form of the Hurewicz theorem since the abelian groups whose elements have bounded orders dividing powers of p form a class (**10**).

THEOREM 1.4. *If A is 1-connected and of homology characteristic $p = p_1 \ldots p_k$, where $p_1, \ldots, p_k$ are powers of distinct primes, then there are complexes $A_1, \ldots, A_k$ of homology characteristics $p_1, \ldots, p_k$, respectively, such that A, $\bigvee_{1 \leqslant t \leqslant k} A_t$, $\prod_{1 \leqslant t \leqslant k} A_t$ have the same homotopy type.*

This is a special case of a theorem concerning spaces whose homology groups are torsion groups. If A, B are 1-connected and of mutually prime homology characteristics, the inclusion $A \vee B \subset A \times B$ induces an homology isomorphism, and hence (by a theorem of J. H. C. Whitehead's) is a homotopy equivalence. Therefore $\bigvee A_t$ and $\prod A_t$ must have the same homotopy type.

† Let $M(Q)$ denote the maximum order of the elements of Q. If $pQ = 0$, both $\mathrm{Hom}(Q, G)$ and $\mathrm{Ext}(Q, G)$ are Z_p modules in an obvious way, and it is easily proved that the maximum orders of their elements divide $M(Q)$, and are $M(Q)$ when $G = Z_p$. Since

$$H^r(A; G) \approx \mathrm{Hom}(H_r(A), G) + \mathrm{Ext}(H_{r-1}(A), G),$$

it follows that $M(H^r(A; G))$ divides $M(H^r(A; Z_p))$ for all G when A has finite homology characteristic p.

An equivalence $A \equiv \prod A_t$ can be constructed by considering the Postnikov system of A; the dual method, described in (9), is to construct a sequence of spaces $B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset B_\infty$, and maps $f_r\colon B_r \to A$ such that

(i) $B_i =$ point,

(ii) $f_n \mid B_{n-1} = f_{n-1}$,

(iii) $f_{n*}\colon H_i(B_n) \approx H_i(A) \quad (i \leqslant n)$,†

(iv) $H_i(B_n) = 0 \quad (i > n)$.

B_n is obtained from B_{n-1} by attaching a cone on a Moore space Y_n by means of a map $Y_n \to B_{n-1}$, where the only non-trivial homology groups of Y_n are

$$H_0(Y_n) = Z, \qquad H_{n-1}(Y_n) = H_n(A).$$

Assume, as an inductive hypothesis, that

$$B_{n-1} = \bigvee B_{n-1,t} \quad (1 \leqslant t \leqslant k),$$

where the homology characteristic of $B_{n-1,t}$ divides p_t. We can also choose Y_n in similar form $\bigvee Y_{n,t}$. By the remark above,

$$\pi_q(B_{n-1}) = \pi_q(\bigvee B_{n-1,t}) \approx \pi_q(\prod B_{n-1,t}) = \sum \pi_q(B_{n-1,t}).$$

By (1.3), all maps $Y_{n,t} \to B_{n-1}$ can be deformed into $B_{n-1,t}$ $(1 \leqslant t \leqslant k)$.‡ Therefore B_n can be constructed so as to be a similar wedge $\bigvee B_{n,t}$. Since B_1 is a point, by the principle of induction

$$B_\infty = \bigcup B_n = \bigvee_{1 \leqslant t \leqslant k} (\bigcup B_{n,t}) \equiv A.$$

This completes the proof.

2. Theorems on suspensions

We first prove the lemma:

LEMMA 2.1. *If A has characteristic p, then*

$$pE\pi_q(A) = 0 \quad (E\colon \pi_q(A) \to \pi_{q+1}(EA)).$$

Proof. The track-group functor $\pi_1^A(X)$ is contravariant in A; a homotopy class $\phi\colon A \to B$ induces a homomorphism $\phi^*\colon \pi_1^B(X) \to \pi_1^A(X)$ by $\phi^*(\theta) = \theta \circ E\phi$. Now $\pi_1^{SA}(X)$ can be identified with $\pi_{q+1}(X)$, so that $\phi^*(\mathbf{1}_A) = E\phi$ if $\phi \in \pi_q(A)$. Therefore $pE\phi = (p\mathbf{1}_A) \circ E\phi = 0$, which proves the lemma.

† The epimorphism when $i = n$ in (9) can be replaced by an isomorphism.

‡ Let C_t be the class of torsion groups whose orders are prime to p_t. Then $H_q(B_{n-1}, B_{n-1,t})$ are in C_t, and so therefore are $\pi_q(B_{n-1}, B_{n-1,t})$. The inclusion $B_{n-1,t} \subset B_n$ therefore induces an isomorphism between $H^r(Y_{n,t}, \pi_q(B_{n-1,t}))$ and $H^r(Y_{n,t}, \pi_q(B_{n-1}))$, and therefore between the homotopy classes of maps of $Y_{n,t}$ into these spaces.

Information about the kernel of E can be obtained from the exact couple of the suspension, described in (**2**), and the following theorems (p denotes an arbitrary prime):

THEOREM 2.2. *If A is $(n-1)$-connected and has characteristic p^m, then*

$$p^{mk}\pi_q(EA) = 0 \text{ for } q \leqslant 2^k n \quad (n \geqslant 1).$$

THEOREM 2.3. *If A in (2.2) is itself a suspension, then*

$$p^{m+k}\pi_q(EA) = 0 \text{ for } q \leqslant p^{k+1}n.$$

The proofs will be given in § 5 and § 7 respectively.

3. Collapsed products and Whitehead products

The collapsed product $A \barwedge B$ of two spaces A, B with base points is the result of identifying $A \vee B$ in $A \times B$ with the base point. This is a functor: maps $\alpha\colon A \to A'$, $\beta\colon B \to B'$ determine $\alpha \barwedge \beta\colon A \barwedge B \to A' \barwedge B'$. Notice that $\barwedge$ is associative, and distributive over $\vee$; S^0 acts as a two-sided unit, and the suspension EA can be defined as $A \barwedge S^1$. The k-fold collapsed product $A^{(k)}$ is defined inductively by

$$A^{(0)} = S^0, \quad A^{(1)} = A, \quad A^{(k+1)} = A \barwedge A^{(k)}.$$

We consider the lemmas:

LEMMA 3.1. *The characteristic of $A \barwedge B$ divides the characteristics of A and B.*

In particular, $A \barwedge B$ has finite characteristic if either A or B has. This will be deduced from the next lemma, which is proved in the Appendix. Let $\iota_A\colon A \to A$ denote the identity map or its homotopy class indifferently. Then every map (or homotopy class) $\beta\colon EB \to X$ determines $\iota_A \barwedge \beta\colon A \barwedge EB = E(A \barwedge B) \to A \barwedge X$.

LEMMA 3.2. *The transformation $\beta \to \iota_A \barwedge \beta$ is a homomorphism*

$$\pi_1^B(X) \to \pi_1^{A \barwedge B}(A \barwedge X).$$

Proof of Lemma 3.1. $\mathbf{1}_A = E\iota_A$,

and $\mathbf{1}_{A \barwedge B} = E(\iota_A \barwedge \iota_B) = \iota_A \barwedge E\iota_B = \iota_A \barwedge \mathbf{1}_B$.

By the lemma, $p(\iota_A \barwedge \mathbf{1}_B) = \iota_A \barwedge (p\mathbf{1}_B) = 0$;

this completes the proof.

It is convenient to define the Whitehead product

$$[\pi_1^A(X), \pi_1^B(X)] \subset \pi_1^{A \barwedge B}(X)$$

by means of commutators [cf. (**1**) § 8.2], generalizing Fox's characterization (**7**), rather than by Cohen's method (**4**); the definitions are

equivalent for suspensions. The natural maps of $A \times B$ to $A \otimes B$, A, B induce monomorphisms by which the groups

$$\pi_1^{A \otimes B}(X), \quad \pi_1^A(X), \quad \pi_1^B(X)$$

can be embedded in $\pi_1^{A \times B}(X)$. It is shown in (**1**) (loc. cit.) that there is a short exact sequence

$$\pi_1^{A \otimes B}(X) \to \pi_1^{A \times B}(X) \xrightarrow{i^*} \pi_1^A(X) + \pi_1^B(X),$$

where i^* is induced by the inclusion $A \vee B \subset A \times B$. Then, if

$$\alpha \in \pi_1^A(X), \qquad \beta \in \pi_1^B(X),$$

the product $[\alpha, \beta]$ is to be the commutator

$$[\alpha, \beta] = \alpha + \beta - \alpha - \beta$$

in $\pi_1^{A \times B}(X)$; it lies in the subgroup $\pi_1^{A \otimes B}(X) = \mathrm{Ker}(i^*)$.

Remark. If $\theta\colon X \to Y$, then $\theta \circ [\alpha, \beta] = [\theta \circ \alpha, \theta \circ \beta]$.

This product has the important property that, if $\theta\colon X \to Y$ is a bundle mapping of a principal fibre bundle with group G, and

$$\Delta\colon \pi_1^A(X) \to \pi_0^A(G)$$

is the transgression homomorphism (always defined), then

$$\Delta[\alpha, \beta] = \langle \alpha, \beta \rangle,$$

where $\langle \alpha, \beta \rangle$ is the Samelson product in $\pi_0^{A \times B}(G)$: for both products are (or can be) defined as commutators. This leads to the statement of the Milnor–Hilton theorem given in § 4.

Another property that will be used in the proof of (2.3) is that of the lemma:

LEMMA 3.3. *If A is a suspension, so is $A \otimes B$,* and $\pi_1^A(X)$, $\pi_1^{A \otimes B}(X)$ *are abelian, while $[\pi_1^A(X), \pi_1^B(X)]$ is linear in $\pi_1^A(X)$. An analogous result applies if B is a suspension.*

The first part is obvious: to prove the linearity, observe that it suffices to show that the subgroup $Q \subset \pi_1^{A \times B}(X)$ generated by $\pi_1^A(X)$, $\pi_1^{A \otimes B}(X)$ is abelian since

$$[\alpha_1 + \alpha_2, \beta] = \alpha_1 + [\alpha_2, \beta] - \alpha_1 + [\alpha_1, \beta].$$

Let $\xi\colon A \times B \to Y$ be the identification map pinching B to a point; then $\xi^*\colon \pi_1^Y(X) \approx Q \subset \pi_1^{A \times B}(X)$, and so $\pi_1^Y(X)$ is to be proved abelian. This will follow if Y is shown to have the homotopy type of some suspension.

Now Y, A, $A \otimes B$ are 1-connected, and the last two are suspensions: it suffices to construct a map $\eta: Y \to A \vee A \otimes B$ that induces a homology isomorphism. Let $\rho: A \to A \vee A$ be the *canonical pinch* as defined in the Appendix: ρ induces $\rho \times 1: A \times B \to (A \vee A) \times B$, and so

$$\rho': Y \to Y \vee Y.$$

The maps $A \times B \to A$, $A \otimes B$ pinch B to a point, and so induce maps $Y \to A$, $A \otimes B$, and hence a map $\lambda: Y \vee Y \to A \vee A \otimes B$; it is easily verified that $\eta = \lambda \circ \rho: Y \to A \vee A \otimes B$ is a suitable map.

Example. Take $X = EA$; then $[\mathbf{1}_A, \mathbf{1}_A] \in \pi_1^{A(2)}(EA)$, and

$$(p\mathbf{1}_A) \circ [\mathbf{1}_A, \mathbf{1}_A] = [p\mathbf{1}_A, p\mathbf{1}_A].$$

When A is a suspension, this is

$$p^2[\mathbf{1}_A, \mathbf{1}_A] = [\mathbf{1}_A, \mathbf{1}_A] \circ (p^2\mathbf{1}_{A^{(2)}}).$$

4. The Milnor-Hilton theorem

In (**8**) Milnor generalized to the loop space of a wedge of suspensions of connected complexes Hilton's theorem (**6**) on the loop space of a wedge of simply-connected spheres. Since the statement of the theorem is not readily available, it is stated here in the form in which it will be used. Milnor's construction is more explicit and can yield more information than the bare statement below.

The theorem concerns the homotopy groups of a wedge of suspensions $X = EA_1 \vee \ldots \vee EA_s$, of connected complexes $A_1, \ldots, A_s$. A countable set $\{A_t\}$ $(t = 1, 2, \ldots)$ of complexes, together with homotopy classes of maps $u_t: EA_t \to X$ will be constructed so that the following theorem is true:

The Milnor–Hilton Theorem. *$u_{t*}: \pi_q(EA_t) \to \pi_q(X)$ is a monomorphism for each $t \geqslant 1$, and*

$$\pi_q(X) = \sum_t u_{t*}\pi_q(EA_t).$$

In fact, π_q can be replaced by the track group π_q^B of a *suspension* B.

Construction of A_t, u_t. Construct first symbolic basic products $\omega_1, \omega_2, \ldots$ in symbols $\mathbf{1}, \ldots, \mathbf{s}$; each ω_t will be a certain number $w_t > 0$ (called the *weight* of ω_t) of symbols, bracketed in some way, and will have a non-negative integer c_t (called the *class* of ω_t) associated with it. The construction is not unique, and proceeds inductively.

There are to be s symbolic basic products $\omega_t = \mathbf{t}$ $(t \leqslant s)$ of weight 1; all have class 0. Assume that the symbolic basic products of weight

less than m have been defined and are $\omega_1, \ldots, \omega_n$, such that

$$\text{(i)} \quad r < t \text{ implies } w_r \leqslant w_t, \qquad \text{(ii)} \quad c_t < t.$$

Take all symbols $[\omega_p, \omega_q]$ such that

$$\text{(i)} \quad w_p + w_q = m, \qquad \text{(ii)} \quad c_q \leqslant p < q;$$

order these arbitrarily and label them $\omega_{n+1}, \omega_{n+2}, \ldots$. These are to be the symbolic basic products of weight m; the class of $[\omega_p, \omega_q]$ is to be p.

Define $A_t = \omega_t(A_1, \ldots, A_s)$ by induction, so that

$$\text{if } t \leqslant s, \; A_t = A_t; \quad \text{if } \omega_t = [\omega_p, \omega_q], \; A_t = A_p \barwedge A_q.$$

Define maps $u_t = \omega_t(\mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_s})$ by

$$\text{if } t \leqslant s, \; u_t = \mathbf{1}_{A_t}; \quad \text{if } \omega_t = [\omega_p, \omega_q], \; u_t = [u_p, u_q].$$

In general, if $\alpha_t \colon EA_t \to X$ $(1 \leqslant t \leqslant s)$ are homotopy classes of maps, $\omega_t(\alpha_1, \ldots, \alpha_s)$ can be similarly defined as a repeated Whitehead product.

The theorem leads to the following generalization of a construction of Hilton's (6). Let $A = A_1 = \ldots = A_s$, and $X = \bigvee_{t \leqslant s} EA_t$ as before. Construct homotopy classes of maps

$$EA \xrightarrow{\rho_s} X \xrightarrow{\mu} EA,$$

so that ρ_s is the track-sum $u_1 + u_2 + \ldots + u_s$ of the s identity maps $EA \to EA_t$, and $\mu \mid EA_t$ is also the identity map $EA_t \to EA$. Thus $\mu \circ \rho_s = (s\mathbf{1}_A)$. Let ρ_* be the homomorphism induced by ρ_s, i.e.

$$\rho_* \colon \pi_q(EA) \to \pi_q(X) = \sum u_{t*}\, \pi_q(EA)_t.$$

By composing ρ_* with the natural projections, natural homomorphisms M_t are obtained:

$$M_t \colon \pi_q(EA) \to \pi_q(EA_t) \quad (t = 1, 2, \ldots) \tag{4.1}$$

such that $\rho_*(\phi) = \sum u_{t*}\, M_t(\phi)$, and hence

$$(s\mathbf{1}_A) \circ \phi = \mu_* \rho_*(\phi) = \sum \mu \circ u_t \circ M_t(\phi). \tag{4.2}$$

Notice that, when $t \leqslant s$,

$$\mu \circ u_t \circ M_t(\phi) = \phi.$$

5. Proof of Theorem 2.2

In (4.2), take $s = p^m$; for any $\phi \in \pi_q(EA)$,

$$0 = (p^m \mathbf{1}_A) \circ \phi = p^m \phi + \sum_{t > p^m} \mu \circ u_t \circ M_t(\phi).$$

Now, for $t > p^m$, A_t is at least a 2-fold collapsed product of A, and so

at least $(2n-1)$-connected and, by Lemma 3.1, of characteristic dividing p^m. Assume the theorem true for all spaces of all prime-power characteristics, all connectivities n, and all $q \leqslant 2^{k-1}n$. Then, if $q \leqslant 2^k n$, $p^{m(k-1)}\pi_q(EA_t) = 0$ for all $t > p^m$. Thus

$$0 = p^{mk}\phi + \sum_{t>p^m} \mu \circ u_t \circ (p^{m(k-1)}M_t(\phi)) = p^{mk}\phi.$$

Therefore the theorem is also true for $q \leqslant 2^k n$; it is trivially true for $q < n$, and so always true. This proves the theorem.

This procedure will be used later, and called 'induction on k'. The next lemma will also be used:

LEMMA 5.1. *If A is a suspension of characteristic p^m, then for each $1 \leqslant r \leqslant m$, $(p^r\mathbf{1}_A) \circ (p^{m-r}\phi) = 0$ for any ϕ in $\pi_q(EA)$ and all q.*

Notice that this gives Theorem 2.3 when $p = 2$, by induction on k: for

$$0 = (p\mathbf{1}_A) \circ (p^{m-1}\phi) = p^m\phi + \sum \mu \circ u_t \circ (p^{m-1}M_t(\phi));$$

on multiplying by p^k and using the appropriate inductive hypothesis, it follows that $p^{m+k}\pi_q(EA) = 0$ for $q \leqslant 2^{k+1}n$. A more delicate argument is required for the other primes.

Proof of Lemma 5.1. The theorem is true when $r = m$. Notice that, if true for any particular r, then

$$(p^{r+t}\mathbf{1}_A) \circ (p^{m-r}\phi) = 0 \quad \text{for all } t \geqslant 0.$$

Assume that, for some $r \geqslant 1$ and all $1 \leqslant s \leqslant m-1$,

$$0 = (p^{r+s}\mathbf{1}_A) \circ (p^{m-r-s}\phi) = 0.$$

Put, for convenience, $\psi = p^{m-r-1}\phi$, and expand $(p^{r+1}\mathbf{1}_A) \circ \psi$ by (4.2) with $s = p$ as follows:

$$\begin{aligned} 0 &= (p^r\mathbf{1}_A) \circ (p\mathbf{1}_A) \circ \psi \\ &= (p^r\mathbf{1}_A) \circ \Big\{ p\psi + \sum_{t>p} \mu \circ u_t \circ M_t(\psi) \Big\} \\ &= (p^r\mathbf{1}_A) \circ p^{m-r}\phi + \Big\{ \sum_{t>p} (p^r\mathbf{1}_A) \circ \mu \circ u_t \circ M_t(\psi) \Big\}. \end{aligned} \tag{5.2}$$

Now

$$(p^r\mathbf{1}_A) \circ \mu \circ u_t = \mu \circ \Big(\bigvee_{i<p} (p^r\mathbf{1}_{A_i}) \circ u_t \Big) = \mu \circ u_t \circ (p^{rw_t}\mathbf{1}_{A_t}),$$

by Lemma 3.4 (cf. example), since A is a suspension.

By the inductive hypothesis (since $w_t > 1$)

$$(p^{rw_t}\mathbf{1}_A) \circ M_t(p^{m-r-1}\phi) = 0 \quad (\text{all } t > p).$$

Therefore (5.2) reduces to $0 = (p^r\mathbf{1}_A) \circ (p^{m-r}\phi)$, and Lemma 5.1 follows by induction on $m-r$.

6. The distributive law

Following Hilton, a distributive law for composition is obtained by taking $s = 2$ in (4.2); in this case, write H_t instead of M_t (conforming with Hilton's notation). If $\alpha, \beta: EA \to Y$, $\alpha \vee \beta: EA \vee EA = X \to Y \vee Y$; let $\theta: X \to Y$ be obtained by collapsing $Y \vee Y$ on Y. Then (4.2) implies

$$(\alpha+\beta) \circ \phi = \theta \circ \rho_2 \circ \phi = \alpha \circ \phi + \beta \circ \phi + \sum_{t>2} \theta \circ u_t \circ H_t(\phi). \quad (6.1)$$

This is the distributive law referred to; the other,

$$\alpha \circ (\phi + \phi') = \alpha \circ \phi + \alpha \circ \phi'$$

is always true and has been used already.

The intention is to expand $(s\mathbf{1}_A) \circ \phi$ by the distributive law; it is necessary to examine $\{(k\mathbf{1}_{A_1}) \vee \mathbf{1}_{A_2}\} \circ u_t$: $EA_t \to EA_1 \vee EA_2$ when A is a suspension.

A symbolic basic product ω_t of weight w_t in symbols $\mathbf{1}$, $\mathbf{2}$ will be said to be of *type* a if there are a $\mathbf{1}$'s and $(w_t - a)$ $\mathbf{2}$'s in it. Let $\alpha = k\mathbf{1}_A$, $\beta = \mathbf{1}_A$: $EA \to EA$ define $\theta: X = EA \vee EA \to EA$ as before. We have

LEMMA 6.2. *$\theta \circ u_t = \mu \circ u_t \circ (k^{a_t}\mathbf{1}_{A_t})$, where ω_t is of type a_t.*

This follows from the definition and Lemma 3.3 (cf. example), by induction on t. Then (6.1) yields, with Lemma 6.2,

$$\{(k+1)\mathbf{1}_A\} \circ \phi - (k\mathbf{1}_A) \circ \phi - \phi = \sum_{t>2} \mu \circ u_t \circ (k^{a_t}\mathbf{1}_{A_t}) \circ H_t(\phi). \quad (6.3)$$

Sum this for $1 \leqslant k < s$ to obtain the lemma:

LEMMA 6.4. $$(s\mathbf{1}_A) \circ \phi - s\phi = \sum_{t>2} \mu \circ u_t \circ \Big\{\sum_{k=1}^{s-1} [(k^{a_t}\mathbf{1}_{A_t}) \circ H_t(\phi)]\Big\}.$$

This is the first step in the reduction of the left-hand side. On expanding the terms in the braces, we obtain invariants of type $H_s H_t(\phi)$, and leave more terms to be expanded. It is convenient to extend the notation temporarily to cover the general case.

Let τ denote an ordered sequence $(t_1, \ldots, t_r)$, where $t_i > 2$, and let T be the set of all such sequences ($r = 1, 2, 3, \ldots$). Define, by induction on r, spaces A_τ, maps $\tilde{u}_\tau$: $EA_\tau \to EA$ and homomorphisms

$$H_\tau: \pi_q(EA) \to \pi_q(EA_\tau)$$

by setting

$$\left.\begin{aligned} A_\tau &= A_t, \quad \tilde{u}_\tau = \mu \circ u_t, \quad H_\tau = H_t, && \text{if } \tau = (t) \\ A_\tau &= (A_t)_\nu, \quad \tilde{u}_\tau = \tilde{u}_t \circ \tilde{u}_\nu, \quad H_\tau = H_\nu \circ H_t, && \text{if } \tau = (t, \nu) \end{aligned}\right\}. \quad (6.5)$$

Remark. These constructions are natural; $\tilde{u}_t \circ \tilde{u}_\nu$ denotes a composition

$$E(A_t)_\nu \to EA_t \to EA.$$

Repetition of Lemma 6.4 yields a formula of type

$$(s\mathbf{1}_A)\circ\phi - s\phi = \sum \bar{u}_\tau \circ \{\sigma_\tau(s) H_\tau(\phi)\},$$

where $\sigma_\tau(s)$ is an integer; it is important to know what these integers are.

Define $\sigma(s;\alpha_1,\ldots,\alpha_r)$ for integers $\alpha_1,\ldots,\alpha_r$ inductively by

$$\sigma(s;\alpha) = \sum_{t=0}^{s-1} t^\alpha, \quad \ldots, \quad \sigma(s;\alpha_1,\ldots,\alpha_r) = \sum_{t=0}^{s-1} \sigma(t^{\alpha_1};\alpha_2,\ldots,\alpha_r). \tag{6.6}$$

If $\tau = (t_1,\ldots,t_r)$, and ω_t is of type a_t, set

$$\sigma_\tau(s) = \sigma(s; a_{t_1},\ldots,a_{t_r}). \tag{6.7}$$

LEMMA 6.8. *If A is a suspension, $\phi \in \pi_q(EA)$, and A_τ, $\bar{u}_\tau$, H_τ, $\sigma_\tau(s)$ are defined by* (6.5), (6.7), *then*

$$(s\mathbf{1}_A)\circ\phi - s\phi = \sum \bar{u}_\tau \circ \{\sigma_\tau(s) H_\tau(\phi)\},$$

the summation being over all τ in T.

Proof. Assume the lemma true for n-connected suspensions, and fixed q; suppose A $(n-1)$-connected. The lemma applies to $\pi_q(EA_t)$ $(t>2)$, and follows for $\pi_q(EA)$ from Lemma 6.4 and the inductive hypothesis; it is trivially true when $n > q$, and so true always.

Remark. Only a finite number of $H_\tau(\phi)$ can be non-zero; also, A_τ is a certain collapsed product of A's, and so a suspension.

LEMMA 6.9. *If $\tau = (t_1,\ldots,t_r)$, and p is a prime greater than the product of the weights $w_{t_1}\ldots w_{t_r}$ of the symbolic basic products $\omega_{t_1},\ldots,\omega_{t_r}$, then p divides $\sigma_\tau(p)$.*

Proof. Since the type a_t of ω_t is less than w_t, it follows that it suffices to prove that p divides $\sigma(p;\alpha_1,\ldots,\alpha_r)$ when $\prod(\alpha_i+1) < p$. Now

$$\sigma(s;\alpha) = \sum_0^{s-1} t^\alpha$$

is a rational polynomial in s without a constant term, and the denominators of the coefficients are products of powers of primes not exceeding the degree $(\alpha+1)$. Such a polynomial will be called *stable*. Assume that $\sigma(s;\alpha_1,\ldots,\alpha_r)$ is a stable polynomial of degree $\prod(\alpha_i+1)$. Then the polynomial

$$\sigma(s;\alpha,\alpha_1,\ldots,\alpha_r) = \sum_{t=0}^{s-1} \sigma(t^\alpha;\alpha_1,\ldots,\alpha_r)$$

is also stable, of degree $(\alpha+1)\prod(\alpha_i+1)$. This proves the lemma since $\sigma(s;\alpha)$ is of degree $(\alpha+1)$.

7. Proof of Theorem 2.3

By (5.1), $$(p\mathbf{1}_A) \circ \psi = 0$$

if $\psi = p^{m-1}\phi$. By (6.8),

$$-p^m\phi = -p\psi = \sum \tilde{u}_\tau \circ \{\sigma_\tau(p)H_\tau(\psi)\}. \tag{7.1}$$

The terms in the braces are of two kinds: if $\tau = (t_1, \dots, t_r)$ and $p > w_{t_1} \dots w_{t_r}$, then p divides $\sigma_\tau(p)$ by Lemma 6.9, while, if $p \leqslant w_{t_1} \dots w_{t_r}$,

$$H_\tau(\psi) \in p^{m-1}\pi_q(EA_\tau),$$

and A_τ is at least $(pn-1)$-connected. In the former case, the term in the braces is a multiple of $p^m H_\tau(\phi)$.

The theorem is trivially true if $q < n$. Assume that it is true for all q and all spaces of connectivity not less than n; suppose that A is $(n-1)$-connected and multiply (7.1) by p^k. The right-hand side becomes zero, and (7.1) becomes $p^{m+k}\phi = 0$. The theorem follows by the principle of induction.

Appendix

To ensure that the product $A \times B$ of complexes is a complex, it is convenient to restrict complexes and maps to the realization of CSS complexes satisfying the extension condition; this imposes no limitation on homotopy type in the category of CW complexes. If A, B are CW complexes with (vertex) base points, $A \otimes B$ is obtained by pinching $A \vee B$ to the base point; S' is the unit circle in the Argand plane, and the (reduced) suspension EA of A can be taken to be $A \otimes S'$. Define the *canonical pinching map*

$$\rho\colon EA \to EA \vee EA$$

as follows. Let $\rho_0\colon S' \to S' \vee S'$ be given by $\rho(e^{2\pi i t}) = e^{4\pi i t}$, with values in the first circle if $0 \leqslant t \leqslant \frac{1}{2}$, and in the second if $\frac{1}{2} \leqslant t \leqslant 1$ (1 being the base point of S'). Then $\rho = \iota_A \otimes \rho_0$, where $\iota_A\colon A \to A$ is the identity map.

The track-sum of maps $\alpha, \beta\colon EA \to X$ is defined as the composition

$$EA \xrightarrow{\rho} EA \vee EA \xrightarrow{\alpha \vee \beta} X \vee X \xrightarrow{\mu} X,$$

where μ is the folding map, the identity on each copy of X; it is easily verified that this induces an associative addition of homotopy classes, and that addition is commutative if A is a suspension, or has the homotopy type of some suspension.

There is a natural one-to-one correspondence between the (base-point preserving) homotopy classes of maps $EA \to X$ and of $A \to \Omega X$, the loop-

space on X. ΩX is an H-space with a homotopy associative multiplication, possessing a homotopy unit and homotopy inverses; it follows that the homotopy classes of maps $A \to \Omega X$ form a group $\pi_0^A(\Omega X)$. The track-addition is such that the correspondence is a homomorphism; thus the set $\pi_1^A(X)$ of homotopy classes of maps $EA \to X$ forms a group under track-addition.

Proof of Lemma 3.2. The naturality of the canonical pinching map ρ implies that

$$\rho = \iota_A \otimes \rho\colon$$

$$E(A \otimes B) = A \otimes EB \to E(A \otimes B) \vee E(A \otimes B) = A \otimes (EB \vee EB).$$

Similarly μ is natural:

$$\mu = \iota_A \otimes \mu\colon A \otimes X \vee A \otimes X = A \otimes (X \vee X) \to A \otimes X.$$

Therefore

$$\begin{aligned} \iota_A \otimes \alpha + \iota_A \otimes \beta &= \mu \circ (\iota_A \otimes \alpha \vee \iota_A \otimes \beta) \circ \rho \\ &= \iota_A \otimes (\mu \circ (\alpha \vee \beta) \circ \rho) = \iota_A \otimes (\alpha + \beta). \end{aligned}$$

REFERENCES

1. M. G. Barratt, Track groups I, II, *Proc. London Math. Soc.* (3) 5 (1955) 71–106; 285–329.
2. —— *Higher Hopf invariants*, University of Chicago mimeographed notes, Summer 1957.
3. M. G. Barratt and J. Milnor, *Anomalous singular homology groups* (to appear).
4. D. E. Cohen, 'Products and carrier theory', *Proc. London Math. Soc.* (3) 7 (1957) 219–48.
5. S. T. Hu, 'Structure of homotopy groups of mapping spaces', *American J. of Math.* 71 (1949) 574–86.
6. P. J. Hilton, 'The homotopy groups of the union of spheres', *J. London Math. Soc.* 30 (1955) 154–72.
7. R. H. Fox, 'Homotopy and torus homotopy groups', *Ann. of Math.* 49 (1948) 471–510.
8. J. Milnor, *The construction FK*, Princeton University mimeographed notes.
9. J. C. Moore, Le théorème de Freudenthal, *Séminaire Henri Cartan* (1954–1955), *Seminar* 22.
10. J.-P. Serre, 'Groupes d'homotopie et classes de groupes abéliens', *Ann. of Math.* 58 (1953) 258–94.
11. J. H. C. Whitehead, 'Combinatorial homotopy I', *Bull. American Math. Soc.* 54 (1948) 1125–32.
12. —— 'A certain exact sequence', *Ann. of Math.* 52 (1950) 51–110.

A MATRIX INEQUALITY

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LET $\{a_{ij}\}$ be an m by n matrix of non-negative terms. Then

$$mn \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} a_{i.} a_{.j} \geqslant (a_{..})^3 \tag{1}$$

where $a_{i.}$ and $a_{.j}$ are the row and column totals and $a_{..}$ is the sum of all the terms. If all the row (or column) sums are equal, the result follows from Cauchy's inequality, and in particular the result is true if either $m = 1$ or $n = 1$.

We say that $A = \{a_{ij}\}$ is *minimal* if

$$\phi = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} a_{i.} a_{.j}$$

attains its minimum for non-negative a_{ij} and $a_{..}$ fixed. The inequality (1) will be proved if we show that for a minimal matrix either all the row sums are equal, or all the column sums are equal, or both. Assume that A is minimal and that at least two of the row sums are unequal and two of the column sums. Assume the matrix rearranged so that

$$a_{1.} \leqslant a_{2.} \leqslant \ldots \leqslant a_{m.}, \tag{2}$$

$$a_{.1} \leqslant a_{.2} \leqslant \ldots \leqslant a_{.n}. \tag{3}$$

We then have $a_{1.} < a_{m.}$ and $a_{.1} < a_{.n}$. It is clearly sufficient to consider the case in which no row or column consists of zeros only, for arguing by induction on $m+n$ it would then follow that (1) would be a strict inequality and A therefore not minimal.

We first show that we may take $a_{1n} > 0$ and $a_{m1} > 0$. Suppose, if possible, that $a_{m1} = 0$. We can then find $a_{r1} > 0$ and $a_{ms} > 0$ since neither the first column nor the last row can consist of zeros only. We make a small perturbation in which

$$a_{r1},\ a_{m1},\ a_{rs},\ a_{ms}$$

are replaced by

$$a_{r1}-\epsilon, \quad a_{m1}+\epsilon, \quad a_{rs}+\epsilon, \quad a_{ms}-\epsilon.$$

The row and column sums are unaltered and the new elements are

non-negative if $\epsilon = \min(a_{r1}, a_{ms})$, whilst a_{m1} is replaced by a positive number. The effect on ϕ is a decrease of amount

$$-\epsilon(a_{r\cdot} - a_{m\cdot})(a_{\cdot 1} - a_{\cdot s})$$

which is certainly not positive. Since A is minimal, ϕ will, in fact, be unaltered. Similarly we may ensure that $a_{1n} > 0$.

We now make another kind of perturbation in which a_{m1}, a_{11}, and a_{1n} are replaced by

$$a_{m1} - x, \quad a_{11} + 2x, \quad a_{1n} - x,$$

where x is small and positive. Taking ϕ as a function of x we shall show that

$$\frac{d\phi}{dx} < 0$$

for $x = 0$, from which it follows that A cannot be minimal, thus giving a contradiction.

The partial derivative of ϕ with respect to an arbitrary element a_{ij} is

$$a_{i\cdot}a_{\cdot j} + \sum_{k=1}^{n} a_{ik}a_{\cdot k} + \sum_{h=1}^{m} a_{hj}a_{h\cdot}.$$

Hence

$$\begin{aligned}\frac{d\phi}{dx} = 2\Big\{&a_{1\cdot}a_{\cdot 1} + \sum_{k=1}^{n} a_{1k}a_{\cdot k} + \sum_{h=1}^{m} a_{h1}a_{h\cdot}\Big\} - \\ &- \Big\{a_{m\cdot}a_{\cdot 1} + \sum_{k=1}^{n} a_{mk}a_{\cdot k} + \sum_{h=1}^{m} a_{h1}a_{h\cdot}\Big\} - \\ &- \Big\{a_{1\cdot}a_{\cdot n} + \sum_{k=1}^{n} a_{1k}a_{\cdot k} + \sum_{h=1}^{m} a_{hn}a_{h\cdot}\Big\}.\end{aligned}$$

Using (2) and (3) we have

$$\sum_{k=1}^{n} a_{1k}a_{\cdot k} \leqslant \sum_{k=1}^{n} a_{1k}a_{\cdot n} = a_{1\cdot}a_{\cdot n},$$

$$\sum_{h=1}^{m} a_{h1}a_{h\cdot} \leqslant \sum_{h=1}^{m} a_{h1}a_{m\cdot} = a_{\cdot 1}a_{m\cdot},$$

$$-\sum_{k=1}^{n} a_{mk}a_{\cdot k} \leqslant -\sum_{k=1}^{n} a_{mk}a_{\cdot 1} = -a_{m\cdot}a_{\cdot 1},$$

$$-\sum_{h=1}^{m} a_{hn}a_{h\cdot} \leqslant -\sum_{h=1}^{m} a_{hn}a_{1\cdot} = -a_{\cdot n}a_{1\cdot}.$$

Hence

$$\frac{d\phi}{dx} \leqslant 2a_{1\cdot}a_{\cdot 1} - a_{m\cdot}a_{\cdot 1} - a_{\cdot n}a_{1\cdot} \leqslant a_{1\cdot}(a_{\cdot 1} - a_{\cdot n}) + a_{\cdot 1}(a_{1\cdot} - a_{m\cdot}) < 0$$

since $a_{1\cdot}$ and $a_{\cdot 1}$ are positive. Hence a minimal matrix makes (1) an equality.

There is an obvious integral analogue to (1). If $K(x, y) \geqslant 0$ is (in

some suitable sense) integrable on the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, it follows by a straightforward argument that

$$ab \int_0^a\int_0^b\int_0^a\int_0^b K(x,t)K(x,y)K(s,y)\,dxdydsdt \geqslant \left\{\int_0^a\int_0^b K(x,y)\,dxdy\right\}^3. \quad (4)$$

If $a = b$ and $K(x,y)$ is symmetric so that $K(x,y) = K(y,x)$, we can write (4) in the form

$$a^2 \int_0^a\int_0^a K_3(x,y)\,dxdy = a^2 \int_0^a\int_0^a\int_0^a\int_0^a K(x,y)K(y,s)K(s,t)\,dxdydsdt$$

$$\geqslant \left\{\int_0^a\int_0^a K(x,y)\,dxdy\right\}^3, \quad (5)$$

where $K_3(x,y)$ is the third-order iterate of $K(x,y)$ in the sense of the theory of iterated kernels of integral equations. This suggests the inequality

$$a^2 \int_0^a\int_0^a K_n(x,y)\,dxdy \geqslant \left\{\int_0^a\int_0^a K(x,y)\,dxdy\right\}^n, \quad (6)$$

which would provide a useful lower bound for the integral of the iterates of a symmetric kernel. We have shown that (6) is true for $n = 3$. For $n = 2$ the analogue of the matrix inequality, with the matrix symmetric, is

$$n \sum_{i,j,k} a_{ij}a_{jk} \geqslant \left(\sum_{ij} a_{ij}\right)^2,$$

which may be written $n \sum_j a_{j\cdot}^2 \geqslant \left(\sum_j a_{j\cdot}\right)^2$ and is therefore clearly true. Similarly (6) is true for $n = 2$. Now suppose that $n = 2^r3^s$, where $r > 0$. Put $n = 2l$. Then

$$a \int_0^a\int_0^a K_n(x,y)\,dxdy \geqslant \left\{\int_0^a\int_0^a K_l(x,y)\,dxdy\right\}^2,$$

and, provided that

$$a^{l-1} \int_0^a\int_0^a K_l(x,y)\,dxdy \geqslant \left\{\int_0^a\int_0^a K(x,y)\,dxdy\right\}^l,$$

it will follow that

$$a^{2l-1} \int_0^a\int_0^a K_n(x,y)\,dxdy \geqslant \left\{\int_0^a\int_0^a K(x,y)\,dxdy\right\}^n.$$

Thus, if the result is true for $n = 2^{r-1}3^s$, it is true for $n = 2^r3^s$. By a similar argument we can show that, if it is true for $n = 2^r3^{s-1}$, it is true for 2^r3^s. Since we have proved it true for $n = 2$, 3, it must be true for all n of the form 2^r3^s, and is probably true for all positive integral values of n, so long as $K(x,y)$ is symmetric.

Returning to the case where symmetry is not required we can derive another version of (1) and (2). If $a_{ij} > 0$ and we have non-negative numbers p_i, q_j such that

$$\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j = 1,$$

then
$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} p_i q_j \sum_{r=1}^{m} a_{rj} p_r \sum_{s=1}^{n} a_{is} q_s \geqslant \Big\{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} p_i q_j\Big\}^3. \tag{7}$$

This can be derived from (1) by approximation using a matrix whose rows are repeated and using rational approximations to the p_i and q_j, or from (4) by putting $K(x, y) = a_{ij}$ when

$$p_0 + \ldots + p_{i-1} \leqslant x < p_0 + \ldots + p_i,$$
$$q_0 + \ldots + q_{j-1} \leqslant y < q_0 + \ldots + q_j,$$

where we make the convention that $p_0 = q_0 = 0$.

If we consider the symmetric version of (7) with $m = n$, $a_{ij} = a_{ji}$ and $p_i = q_i$, we obtain an inequality conjectured by Mandel and Hughes (1) on genetical grounds.

Suppose that we have a genetic population of diploid individuals and we consider a single locus at which there are n alleles $A_1, \ldots, A_n$. Let $a_{ij} \geqslant 0$ be the relative viability of the genotype $A_i A_j$ and p_i the frequency of A_i in a particular generation. The mean viability of the population is

$$V = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} p_i p_j$$

if there is random mating. The frequency p_i' of A_i in the next generation is given by the equation

$$V p_i' = p_i \sum_{j=1}^{n} a_{ij} p_j,$$

and the mean viability in the next generation is

$$V' = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} p_i' p_j'.$$

We expect on biological grounds that $V' \geqslant V$, and this is equivalent to (7) with $m = n$, $a_{ij} = a_{ji}$, and $p_i = q_i$.

REFERENCE

1. S. P. H. Mandel and I. M. Hughes, *Nature*, 182 (1958) 63–64.

[*Note added in proof*] If in (1) we take $m = n$, $a_{ij} = a_{ji}$, we get a special case of the inequality proved by H. P. Mulholland and C. A. B. Smith [*American Math. Monthly* 66 (1959) 673–83], which appeared after this paper was written, and which has affinities with our conjecture (6).

ON INFINITELY GENERATED MODULES

By E. J. BRODY (*Singapore*)

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1. Introduction

THE classical theory of elementary ideals and elementary divisors of finitely generated modules can be used to calculate topological invariants of polygonal knots and links (**6**) from presentations of their fundamental groups. In this paper I shall present a generalized elementary divisor theory which applies to infinitely generated modules over a unique factorization domain. This and the techniques of the free calculus (**2**) and (**3**) will be used to calculate new topological invariants of wildly imbedded knots. As illustrations I shall discuss certain knots related to the wild arcs of Fox and Artin (**4**) and (**5**).

I am indebted to Professor R. H. Fox for his help in the preparation of this paper.

2. Divisor topologies and nullity

Let us first review the classical elementary divisors. Let $\mathfrak{M}$ be a finitely generated (abbreviated f.g.) module over a unique factorization domain P. Then $\mathfrak{M}$ is the homomorphic image of a free P-module generated by $\mathfrak{x}_1,\ldots,\ \mathfrak{x}_n$ $(n < \infty)$; the kernel of the homomorphism is generated by the elements

$$\alpha_{i1}\mathfrak{x}_1+\ldots+\alpha_{in}\mathfrak{x}_n \quad (i = 1,\ldots,m;\ m \leqslant \infty).$$

Let A be the $m\times n$ *relation matrix* $[\alpha_{ij}]$. *If* $0 \leqslant k \leqslant n-1$, *the* k*-th elementary divisor* $\Delta_k(\mathfrak{M})$ *is that principal ideal for which a generating element is the greatest common divisor* (g.c.d.) *of the minor determinants of* A *of order* $n-k$. *If* $k \geqslant n$, *we define* $\Delta_k(\mathfrak{M}) = P$. For each $k \geqslant 0$, $\Delta_k(\mathfrak{M})$ is an invariant of the module $\mathfrak{M}$.

[2.1] $\Delta_j(\mathfrak{M})$ *divides* $\Delta_k(\mathfrak{M})$ *for* $j \geqslant k$.

Proof. It follows from the Laplace expansion that each of the determinants defining $\Delta_k(\mathfrak{M})$ is a linear combination of the determinants defining $\Delta_j(\mathfrak{M})$.

[2.2] *If* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are f.g. P-modules and* $\mathfrak{N}$ *is a P-homomorphic image of* $\mathfrak{M}$, *then* $\Delta_k(\mathfrak{N})$ *divides* $\Delta_k(\mathfrak{M})$ *for all* $k \geqslant 0$.

Proof. A relation matrix $A(\mathfrak{N})$ can be obtained from a relation matrix $A(\mathfrak{M})$ by adjoining rows. For any $k \geqslant 0$, $A(\mathfrak{N})$ has more minors of order $n-k$ than has $A(\mathfrak{M})$.

[2.3] *Let $\mathfrak{N}$ be a f.g. submodule of a f.g. P-module $\mathfrak{M}$. Then $\Delta_k(\mathfrak{N})$ divides $\Delta_k(\mathfrak{M})$ for all $k \geqslant 0$.*

The proof can be found in [(**1**) 349].

Consider now an arbitrary P-module $\mathfrak{M}$. If $\mathfrak{M}_\alpha$, $\mathfrak{M}_\beta$ are f.g. submodules of $\mathfrak{M}$, then $\mathfrak{M}$ has a f.g. submodule $\mathfrak{M}_\gamma$ which contains both $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta$. It follows by [2.3] that, for any $k \geqslant 0$, the collection of ideals $\Delta_k(\mathfrak{M}_\alpha)$, taken as a system of neighbourhoods of zero in P, defines a topology† for P. We call this the *k-th divisor topology* $\nabla_k(\mathfrak{M})$.

Let σ and τ be topologies on a set P. We write $\tau \succ \sigma$ if every open set of σ is an open set of τ. Consider the totality of quotient modules $\mathfrak{M}/\mathfrak{M}_\alpha$, where $\mathfrak{M}_\alpha$ is f.g. We define the *k-th residual topology* $\Gamma_k(\mathfrak{M})$ *as the supremum‡ (with respect to the ordering $\succ$) of the topologies* $\nabla_k(\mathfrak{M}/\mathfrak{M}_\alpha)$. It follows directly from Theorem 2.3 that

[2.4] *If $\mathfrak{N}$ is a submodule of $\mathfrak{M}$, then*

$$\text{(i)} \quad \nabla_k(\mathfrak{M}) \succ \nabla_k(\mathfrak{N}),$$

$$\text{(ii)} \quad \Gamma_k(\mathfrak{M}) \succ \Gamma_k(\mathfrak{N})$$

for all $k \geqslant 0$.

[2.5] *If $h: \mathfrak{M} \to \mathfrak{N}$ is a P-homomorphism onto, then*

$$\text{(i)} \quad \nabla_k(\mathfrak{M}) \succ \nabla_k(\mathfrak{N}),$$

$$\text{(ii)} \quad \Gamma_k(\mathfrak{M}) \succ \Gamma_k(\mathfrak{N})$$

for all $k \geqslant 0$.

Proof. Every f.g. submodule $\mathfrak{N}_\alpha$ of $\mathfrak{N}$ is of the form $h(\mathfrak{M}_\alpha)$ for some f.g. $\mathfrak{M}_\alpha$. Then, by [2.2], $\Delta_k(\mathfrak{N}_\alpha)$ divides $\Delta_k(\mathfrak{M}_\alpha)$. This proves (i). Since $\mathfrak{N}/\mathfrak{N}_\alpha$ is a P-homomorphic image of $\mathfrak{M}/\mathfrak{M}_\alpha$, it then follows that

$$\nabla_k(\mathfrak{M}/\mathfrak{M}_\alpha) \succ \nabla_k(\mathfrak{N}/\mathfrak{N}_\alpha).$$

This proves (ii).

If $\mathfrak{M}$ is f.g., then $\nabla_k(\mathfrak{M})$ is determined by $\Delta_k(\mathfrak{M})$, the smallest neighbourhood of zero in the system. Furthermore, two elements of P are contained in disjoint open sets of $\nabla_k(\mathfrak{M})$ if and only if they lie in distinct cosets of the ideal $\Delta_k(\mathfrak{M})$. Therefore

† After adjoining P and the null set, if necessary. This topology need not be Hausdorff, although it obviously makes the ring operations in P continuous.

‡ i.e. the open sets of the *supremum* of a collection of topologies on P are those subsets of P which are open in each topology of the collection.

[2.6] *If* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *are f.g. P-modules, then* $\nabla_k(\mathfrak{M}) = \nabla_k(\mathfrak{M}')$ *if and only if* $\Delta_k(\mathfrak{M}) = \Delta_k(\mathfrak{M}')$.

Thus, in the f.g. case the notion of kth-divisor topology effectively coincides with the classical notion of kth elementary divisor. It is interesting, however, that the concept of residual topology Γ_k becomes trivial in the f.g. case. For then Γ_k is the *trivial topology* (i.e. consists only of P itself and the null set) for all $k \geqslant 0$. That the definition need not be trivial for infinitely generated modules is shown by the following simple example.

Let P be the integers and let $\mathfrak{M}$ the direct sum of an infinite number of cyclic groups of order p and a finite number n of cyclic groups of order q. Then $\Gamma_k(\mathfrak{M})$ is the p-adic topology for all $k \geqslant 0$, while $\nabla_0(\mathfrak{M})$, say, is defined by the sequence of ideals $(q^n p^i)$ $(i = 1, 2, \ldots)$.

Recall that the *nullity* $\nu(\mathfrak{M})$ of a f.g. P-module $\mathfrak{M}$ is the least integer $\nu \geqslant 0$ such that $\Delta_\nu(\mathfrak{M}) \neq (0)$. We define the *nullity* $\nu(\mathfrak{M})$ of an arbitrary P-module as the supremum of $\nu(\mathfrak{M}_\alpha)$, taken over all f.g. submodules $\mathfrak{M}_\alpha$ of $\mathfrak{M}$. If $\mathfrak{M}$ is f.g., then by [2.6] the definitions agree.

3. Computation

Let $\mathfrak{M}$ be a countable P-module. Then there is a sequence

$$\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \ldots$$

of f.g. submodules with $\mathfrak{M} = \bigcup \mathfrak{M}_i$. If $\mathfrak{M}_\alpha$ is any f.g. submodule of $\mathfrak{M}$, then there is an i such that $\mathfrak{M}_\alpha \subset \mathfrak{M}_i$. Hence, by [2.2] and [2.5], $\nabla_k(\mathfrak{M})$ and $\Gamma_k(\mathfrak{M})$ are determined, respectively, by the sequences $\Delta_k(\mathfrak{M}_i)$ and $\nabla_k(\mathfrak{M}/\mathfrak{M}_i)$ $(i = 1, 2, \ldots)$.

I shall make three remarks concerning the computation of these invariants from a relation matrix. The following notation will be used: $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots\}$ will mean that the module $\mathfrak{M}$ is generated by $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$, and ${}_n\mathfrak{M}$ will denote the submodule of $\mathfrak{M}$ generated by $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$. Of course, ${}_n\mathfrak{M}$ is not determined by n and $\mathfrak{M}$ alone, but by a given choice and ordering of generators of $\mathfrak{M}$. An analogous notation will be used in § 4 for groups.

[3.1] *If* P *is Noetherian,* $\mathfrak{X} = \{\mathfrak{x}_1, \mathfrak{x}_2, \ldots\}$ *a* P*-module, and* $\mathfrak{R}$ *a submodule of* $\mathfrak{X}$*, then there exists a set of generators* $\mathfrak{r}_1, \mathfrak{r}_2, \ldots$ *for* $\mathfrak{R}$ *and a sequence* $i(1) \leqslant i(2) \leqslant \ldots$ *of positive integers such that* $\mathfrak{R} \cap {}_n\mathfrak{X}$ *is generated by*

$$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_{i(n)} \quad (n = 1, 2, \ldots).$$

Proof. $\mathfrak{R} \cap {}_1\mathfrak{X}$ is generated by a finite set $\mathfrak{r}_1, \ldots, \mathfrak{r}_{i(1)}$. Assume that $\mathfrak{r}_1, \ldots, \mathfrak{r}_{i(n-1)}$ have been chosen. Then $\mathfrak{R} \cap {}_n\mathfrak{X}$ has a finite set of generators

which may be supposed to contain $\mathfrak{r}_1, \ldots, \mathfrak{r}_{i(n-1)}$; write this set as $\mathfrak{r}_1, \ldots, \mathfrak{r}_{i(n-1)}, \ldots, \mathfrak{r}_{i(n)}$. Since $\mathfrak{R} = \bigcup (\mathfrak{R} \cap {}_n\mathfrak{X})$, we have $\mathfrak{R} = \{\mathfrak{r}_1, \mathfrak{r}_2, \ldots\}$ as required.

Any countable P-module $\mathfrak{M}$ is the image of a countable free P-module $\mathfrak{X}$ under a P-homomorphism ϕ with kernel $\mathfrak{R}$. Thus [3.1] merely states that the corresponding relation matrix for $\mathfrak{M}$ can be given a form in which the first n columns and first $i(n)$ (truncated) rows constitute a relation matrix for the submodule $\phi({}_n\mathfrak{X})$. The following is often useful as a sufficient condition for a given matrix to be of this form:

[3.2] *Let $[\alpha_{ij}]$ be a relation matrix for $\mathfrak{M} = \mathfrak{X}/\mathfrak{R}$ and let $i(1) \leqslant i(2) \leqslant \ldots$ be a sequence of positive integers such that*

(i) $\alpha_{1j} = \alpha_{2j} = \ldots = \alpha_{i(n)j} = 0$ *for* $j > n$,

(ii) *the row vectors $\mathfrak{r}_{i(n)+1}, \ldots, \mathfrak{r}_{i(n+1)}$ are linearly independent modulo ${}_n\mathfrak{X}$.*

Then the first n columns and first $i(n)$ rows constitute a relation matrix for ${}_n\mathfrak{M} = \{\phi(\mathfrak{x}_1), \ldots, \phi(\mathfrak{x}_n)\}$.

Proof. Let ${}_{i(n)}\mathfrak{R} = \{\mathfrak{r}_1, \ldots, \mathfrak{r}_{i(n)}\}$. Then condition (i) means that

$$ {}_{i(n)}\mathfrak{R} \subset \mathfrak{R} \cap {}_n\mathfrak{X}. $$

Condition (ii) states that ${}_{i(n)}\mathfrak{R} \supset \mathfrak{R} \cap {}_n\mathfrak{X}$. Hence

$$ {}_n\mathfrak{M} \approx {}_n\mathfrak{X}/(\mathfrak{R} \cap {}_n\mathfrak{X}) = {}_n\mathfrak{X}/{}_{i(n)}\mathfrak{R}, $$

and the assertion is proved.

In the applications to knot theory the relation matrix often takes the doubly infinite form

$$ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & A_{n-1} & B_{n-1} & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & A_n & B_n & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 & A_{n+1} & B_{n+1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} $$

where A_i, B_i ($i = 0, \pm 1, \pm 2, \ldots$) are square matrices with non-vanishing determinants and the entries outside of the A_i, B_i are zero. The columns of A_n correspond to generators $\mathfrak{x}_{n1}, \ldots, \mathfrak{x}_{nk}$, say, of the free module $\mathfrak{X}$. In this situation we have

[3.3] *The submatrix $[A_n\, B_n]$ is a relation matrix for the submodule of $\mathfrak{M}$ generated by $\phi(\mathfrak{x}_{n1}), \ldots, \phi(\mathfrak{x}_{nk})$, $\phi(\mathfrak{x}_{n+1,1}), \ldots, \phi(\mathfrak{x}_{n+1,k})$.*

Proof. Suppose that, for some fixed n, there is a linear combination of row vectors having only vanishing entries outside the columns of A_n, B_n. Let r be the greatest integer such that the rows of A_r, B_r are

non-trivially involved in this combination. If $r > n$, then the entries of this combination in the columns of B_r must vanish. This would imply that the corresponding non-trivial linear combination of the rows of B_r vanishes, which is impossible since $|B_r| \neq 0$. A similar argument, using $|A_r| \neq 0$, shows that the smallest such integer r' cannot be less than n. Hence the linear combination in question involves only the rows of A_n, B_n.

4. The Alexander module of a group

The notation of this section is essentially that of Fox (**2**) and (**3**).

We first introduce a straightforward generalization of the Tietze transformations. Let $(\mathbf{x}:\mathbf{r})$ be a (*not necessarily finitely generated*) group presentation. Then operation (0) permutes the indices of the generators $(\mathbf{x})$ or of the relators $(\mathbf{r})$. Operation (I) adjoins to the set $(\mathbf{r})$ any set of consequences of $(\mathbf{r})$. Operation (II) adjoins to the set $(\mathbf{x})$ a set $(\mathbf{y})$ and simultaneously adjoins to $(\mathbf{r})$ a set $(s_y \mid y \in (\mathbf{y}))$, where s_y is of the form yu_y^{-1} and u_y is a word in X.

THEOREM 4.1. *If the presentations* $(\mathbf{x}:\mathbf{r})$ *and* $(\mathbf{y}:\mathbf{s})$ *define isomorphic groups, then it is possible to pass from one to the other by a finite sequence of operations* (0), (I), (II) *and their inverses.*

Proof. We argue as in Fox [(**3**) Theorem 1.1], but allow the ranges of the appropriate indices to be infinite.

In a parallel manner we introduce a generalization of the elementary matrix transformations (**7**) and (**3**). Let P be a ring with unit and $A = [\alpha_{ij}]$ a matrix over P with an (possibly) infinite number of rows and columns, i.e. i and j may range over sets of arbitrary cardinality. The elementary transformations are as follows:

(0) permute the rows of A in any way, or permute the columns of A in any way,

(I) adjoin to A any set of rows such that each new row is a left linear combination of the rows of A,

(II) adjoin to A a set of rows r_β and columns c_β such that the intersection of r_β and c_β is the identity 1 of P, all other entries of c_β are zero, and all other non-zero entries of r_β lie in the original columns of A.

[4.2] *Let A be a relation matrix for the left P-module $\mathfrak{M}_A$. Then the operations* (0), (I), *and* (II) *do not change the isomorphism type of $\mathfrak{M}_A$.*

Proof. We follow Zassenhaus [(**7**) 88].

Now let $\phi: X \to G$ be the homomorphism associated with a presenta-

tion $(\mathbf{x}:\mathbf{r})$ of a group G and let $\psi: G \to H$ be a homomorphism onto. Let JH be the integral group ring of H, and denote by $\mathfrak{d}^{\psi\phi}X$ the free left JH-module over the symbols $\mathfrak{d}x_j$ corresponding to the generators x_j of $(\mathbf{x})$. If Y is a subset of X, then $\mathfrak{d}^{\psi\phi}Y$ will denote the submodule of $\mathfrak{d}^{\psi\phi}X$ generated by the elements

$$\mathfrak{d}^{\psi\phi}y = \sum_j (\partial y/\partial x_j)^{\psi\phi}.\mathfrak{d}x_j \quad (y \in Y).$$

The *Jacobian module* $\mathfrak{d}^{\psi}G$ is the quotient module $\mathfrak{d}^{\psi\phi}X/\mathfrak{d}^{\psi\phi}R$; its relation matrix $[(\partial r_i/\partial x_j)]^{\psi\phi}$ is called the *Jacobian matrix* at ψ of the presentation. It is easily verified that the generalized Tietze transformations (0), (I), (II) of $(\mathbf{x}:\mathbf{r})$ induce elementary transformations (0), (I), (II), respectively, of the Jacobian matrix. Hence by [4.1] and [4.2] the isomorphism type of $\mathfrak{d}^{\psi}G$ is independent of the presentation $(\mathbf{x}:\mathbf{r})$.

The *Betti group* B of G is the factor group of the commutator quotient group G/G_2 by its torsion subgroup. If $\psi: G \to B$ is the natural homomorphism, then we call $\mathfrak{d}^{\psi}G$ and $[(\partial r_i/\partial x_j)^{\psi\phi}]$ the *Alexander module* and *Alexander matrix*, respectively. Henceforth we shall suppose that B *is free abelian*, so that JB is essentially a polynomial ring, and therefore §§ 2, 3 apply. We then define the *k-th divisor topology* $\nabla_k(G)$, *k-th residual topology* $\Gamma_k(G)$ and *nullity* $\nu(G)$ as the kth divisor topology, kth residual topology, and nullity, respectively, of the Alexander module $\mathfrak{d}^{\psi}G$. They are clearly invariants of G. More precisely:

[4.3] *If G and G' are isomorphic groups, then there exists an isomorphism $\theta: B \to B'$ of their Betti groups such that the open sets of $\nabla_k(G')$ and $\Gamma_k(G')$ are the images, under the induced isomorphism $\theta: JB \to JB'$, of the open sets of $\nabla_k(G)$ and $\Gamma_k(G)$, respectively, for all $k \geqslant 0$.*

[4.4] *Let G, G' be groups with $B = B'$ and let θ be a homomorphism of G onto G' such that $\psi'\theta = \psi$. Then $\nabla_k(G) \succ \nabla_k(G')$ and $\Gamma_k(G) \succ \Gamma_k(G')$ for all $k \geqslant 0$.*

Proof. If $(\mathbf{x}:\mathbf{r})$ is a presentation of G with associated homomorphism ϕ, then for some set $(\mathbf{r}') \subset X$ there is a presentation $(\mathbf{x}:\mathbf{r},\mathbf{r}')$ of G' with associated homomorphism $\phi' = \theta\phi$. Hence

$$\mathfrak{d}^{\psi'}G' = \mathfrak{d}^{\psi'\phi'}X/\mathfrak{d}^{\psi'\phi'}(\mathbf{r} \cup \mathbf{r}') = \mathfrak{d}^{\psi\phi}X/\{\mathfrak{d}^{\psi\phi}(\mathbf{r}) + \mathfrak{d}^{\psi\phi}(\mathbf{r}')\},$$

i.e. $\mathfrak{d}^{\psi'}G'$ is a JB-homomorphic image of $\mathfrak{d}^{\psi}G$. The conclusions then follow from [2.5].

[4.5] *Suppose that G/G_2 has no elements of finite order, so that $G/G_2 = B$. Then $\nabla_0(G)$ depends only upon B. It is discrete if B is non-trivial, and trivial if B is trivial.*

Proof. The conclusions can be verified by direct computation in the case when $G = B$, a free abelian (or trivial) group. When $G \neq B$, a presentation of B can be obtained from a presentation $(\mathbf{x}:\mathbf{r})$ of G by adjoining to $(\mathbf{r})$ the commutators $c_{ij} = [x_i, x_j]$ for all x_i, x_j of $(\mathbf{x})$. If B is trivial, then $(\partial c_{ij}/\partial x_k)^{\psi\phi} = 0$ for all i, j, k, so that the Alexander matrix of B is substantially the same as that of G. If B is non-trivial, then $\nabla_0(G) > \nabla_0(B)$ by [4.4] and, since $\nabla_0(B)$ is discrete, $\nabla_0(G)$ is discrete. This completes the proof.

As a corollary, we have

[4.6] *The nullity of a knot group is not less than* 1.

However, since the Alexander polynomial of a polygonal knot of one component has coefficient-sum one, it follows from [2.6] that

[4.7] *A knot of one component having a group of nullity greater than* 1 *is necessarily wild* (**5**).

Let σ be a topology for a ring P, and suppose that σ is defined by the collection of ideals I_α. Let $p \in P$. Then we denote by $p\sigma$ the topology defined by the ideals pI_α.

[4.8] *Let A be an Alexander matrix of a group G whose Betti group B is of finite rank. Let $\nabla^{(j)}$ be the 0-th divisor topology of the submatrix obtained by deleting the j-th column of A and let*

$$\delta = \text{g.c.d.}(b-1) \quad (b \in B).$$

Then, if $x_j \in (\mathbf{x})$ is the generator corresponding to the j-th column of A,

$$\nabla^{(j)} = \{(x_j-1)^{\psi\phi}/\delta\}.\nabla_1(G).$$

Proof. Let ${}_n\mathfrak{d}^\psi G$ denote the submodule of $\mathfrak{d}^\psi G$ generated by the images of $\mathfrak{d}x_1, \ldots, \mathfrak{d}x_n$ under the natural homomorphism $\mathfrak{d}^{\psi\phi}X \to \mathfrak{d}^\psi G$. By [3.1] we may suppose that, for all $n \geqslant 1$, there is a submatrix ${}_nA$ of A which is a relation matrix for ${}_n\mathfrak{d}^\psi G$. Denote by ξ_j the jth column of A. If $n \geqslant j$, let ${}_{n\alpha}\Delta^{(j)}$ be the determinant of any set α of $n-1$ rows of ${}_nA - \xi_j$. Then it follows [(**6**) § 3] that, for $n \geqslant j, k$,

$${}_{n\alpha}\Delta^{(j)}.(x_k-1)^{\psi\phi} = (-1)^{j-k}\,{}_{n\alpha}\Delta^{(k)}.(x_j-1)^{\psi\phi}.$$

If ${}_n\Delta^{(j)} = \text{g.c.d.}_\alpha({}_{n\alpha}\Delta^{(j)})$, then

$${}_n\Delta^{(j)}.(x_k-1)^{\psi\phi} = {}_n\Delta^{(k)}.(x_j-1)^{\psi\phi},$$

and so $(x_j-1)^{\psi\phi}$ divides ${}_n\Delta^{(j)}.(x_k-1)^{\psi\phi}$ $(k = 1, \ldots, n)$. Hence also $(x_j-1)^{\psi\phi}$ divides ${}_n\Delta^{(j)}.\delta_n$, where

$$\delta_n = \text{g.c.d.}(x_i-1)^{\phi\psi} \quad (i = 1, \ldots, n).$$

Since B is of finite rank, $\delta_n = \delta$ for sufficiently large n. Therefore

$$({}_n\Delta^{(j)}.\delta)/(x_j-1)^{\psi\phi} = ({}_n\Delta^{(k)}.\delta)/(x_k-1)^{\psi\phi}.$$

Denote by ${}_n\Delta$ the common value of

$$({}_n\Delta^{(j)}.\delta)/(x_j-1)^{\psi\phi} \quad (j = 1,\ldots,n).$$

Then
$${}_n\Delta^{(j)} = \{(x_j-1)^{\psi\phi}/\delta\}.{}_n\Delta. \tag{a}$$

Since
$${}_n\Delta = \text{g.c.d.}({}_n\Delta^{(j)}) \quad (j = 1,\ldots,n),$$

the ideals $({}_n\Delta)$ $(n = 1, 2,\ldots)$ define the topology ∇_1. The ${}_n\Delta^{(j)}$ $(n = j, j+1,\ldots)$ define $\nabla^{(j)}$. Therefore the conclusion follows from equation (a).

A useful corollary is

[4.9] *For the group of a knot of one component,*

$$\nabla^{(j)} = \{(x_j-1)^{\psi\phi}/(t-1)\}.\nabla_1(G),$$

where t is a generator of the Betti group. For a knot of more than one component,

$$\nabla^{(j)} = (x_j-1)^{\psi\phi}.\nabla_1(G).$$

5. Examples

The terms *k-th divisor topology*, *k-th residual topology*, and *nullity* of a *knot* will refer to the fundamental group of its complementary space. Computation is easily carried out by the methods of § 2 if the Alexander matrices are first simplified by generalized elementary transformations. Details are omitted.

Consider the ellipsoid of revolution E of [(**5**) Fig. 2] dissected into the two end points p, q of its axis of revolution and a doubly infinite sequence

$$\ldots, f_{-1}(C), f_0(C), f_1(C),\ldots$$

of cross-sectional slabs. Let L be a simple polygonal arc in the xz-plane with endpoints p and q and interior disjoint from E.

EXAMPLE 5.1. *A wild knot of nullity one.*

Using the notation of (**5**), let L' be a simple polygonal arc in the cylinder C joining s_- and s_+ in the manner indicated in Fig. 1. The set $L \cup \bigcup_{n=-\infty}^{\infty} f_n(L')$ is then a simple closed curve of nullity one, and ∇_1 is defined by the ideals $(1-t+t^2)^i$ $(i = 1, 2,\ldots)$. This, then, is the result one would intuitively expect from a 'doubly infinite product' of trefoil knots.

EXAMPLE 5.2. *A knot of nullity two.*

This is the simple closed curve described in (**4**); ∇_2 is trivial and Γ_0 is defined by the ideals $(t-1)^i$ $(i = 1, 2,\ldots)$.

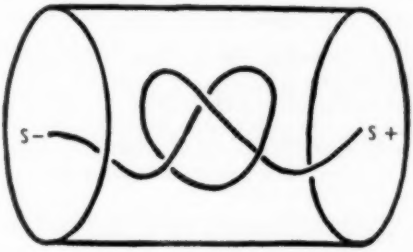

FIG. 1

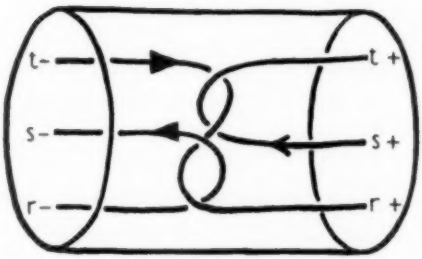

FIG. 2

EXAMPLE 5.3. *A knot of nullity three.*

Referring again to (**5**) let L' be the union of three simple arcs arranged as indicated in Fig. 2. The set $L \cup \bigcup_{n=-\infty}^{\infty} f_n(L')$ can be shown to have the desired property.

By combining the knots of these examples as finite or infinite products, one can construct knots of arbitrary finite or countably infinite nullity.

6. Open questions

In all the examples of wild knots which I have examined the nullity of the knot seems to be equal to the inferior limit of the genera of any sequence of tubular figures (Henkelkörper) approximating to the knot. Does the nullity, then, have a simple geometrical interpretation?

If G/G_2 is trivial, then [(**3**) (3.5)] the Alexander module is trivial. Can the methods of this paper be so modified as to become applicable to wild simple *arcs* in a non-trivial fashion?

REFERENCES

1. R. C. Blanchfield, 'Intersection theory of manifolds with operators', *Ann. of Math.* 65 (1957) 340–56.
2. R. H. Fox, 'Free differential calculus I', ibid. 57 (1953) 547–60.
3. —— 'Free differential calculus II', ibid. 59 (1954) 196–210.
4. —— 'A remarkable simple closed curve', ibid. 50 (1949) 264–5.
5. R. H. Fox and E. Artin, 'Some wild cells and spheres in three-dimensional space', ibid. 49 (1948) 979–90.
6. G. Torres, 'On the Alexander polynomial', ibid. 57 (1953) 57–89.
7. H. Zassenhaus, *Theory of groups* (New York, 1949).

SOME THEOREMS ON MERCER'S AND OTHER RELATED TRANSFORMATIONS

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1. For any complex number λ, let H_λ denote the (clearly regular) transformation from the sequence $\{s_n\}$ to the sequence $\{u_n\}$ given by

$$u_n = \lambda s_n + (1-\lambda)\sigma_n, \tag{1}$$

where

$$\sigma_n = \frac{1}{n+1}(s_0+s_1+\ldots+s_n). \tag{2}$$

It is known† that, if $\mathrm{re}\,\lambda > 0$, then H_λ is equivalent to convergence. If $\mathrm{re}\,\lambda < 0$ and if $u_n \to s$ as $n \to \infty$, then, for large n,

$$s_n = K\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} + s + o(1), \tag{3}$$

where K is a constant, and where we write‡ α for $1/\lambda$. Thus if, in this case, in addition to supposing that $\{u_n\}$ converges, we suppose (for example) that the Abel means of $\{s_n\}$ are bounded, it follows that $\{s_n\}$ converges. This result becomes false when $\mathrm{re}\,\lambda = 0$. Its falsity may be shown by the example§

$$s_0 = 1, \qquad s_n = \frac{n!(1-\alpha)}{(\alpha+1)(\alpha+2)\ldots(\alpha+n)} \quad (n \geqslant 1).$$

Here $u_n = 0$ for $n \geqslant 1$, and s_n is itself bounded, so that, *a fortiori*, its Abel means are bounded. However, if, instead of supposing only that the Abel means of $\{s_n\}$ are bounded, we suppose that $\{s_n\}$ is Abel-summable, the result remains true when $\mathrm{re}\,\lambda = 0$, $\lambda \neq 0$. In other words, we have the theorem:

THEOREM 1. *Suppose that* $\lambda \neq 0$. *Suppose that* $\{s_n\}$ *is summable* H_λ, *and that it is also Abel-summable. Then* $\{s_n\}$ *is convergent.*

We do not need to *assume* that the Abel limit of $\{s_n\}$ is the same as its H_λ limit; this *follows* from the conclusion. As has already been indicated, when $\mathrm{re}\,\lambda \neq 0$, the conclusion of the theorem follows at once from known results (and *more* is true). We may therefore confine our attention to the case in which $\mathrm{re}\,\lambda = 0$.

† Hardy (**2**), Theorem 52, with an obvious change of notation, the λ of the present paper corresponding to Hardy's $1/(1-a)$.

‡ This notation will be used throughout the paper.

§ We assume that $\lambda \neq 0$.

Some of the interest of this theorem lies in its connexion with a recent paper by Jakimovski and Parameswaran (3). For any real κ, let us write Θ^κ for the Hölder transformation (H,κ); let us write Θ for Θ^1, and I for the identical transformation. The transformation

$$I-\mu\Theta, \tag{4}$$

apart from a constant factor, is the same as H_λ with $\lambda = 1/(1-\mu)$; thus Theorem 1 can be expressed in terms of the transformation (4). Jakimovski and Parameswaran observed that the case $\operatorname{re}\lambda \neq 0$ of Theorem 1 could be deduced from known results, and, by means of repeated applications of this theorem, expressed in terms of the transformation (4), proved the theorem:†

THEOREM A. *Suppose that the (possibly complex) constants $c_1, c_2, \ldots, c_r$ satisfy 'the condition (γ)': that is to say, that*

$$P(t) = t^r + c_1 t^{r-1} + c_2 t^{r-2} + \ldots + c_r$$

has no zeros μ satisfying $\operatorname{re}\mu = 1$, $\mu \neq 1$. Suppose that the sequence $\{s_n\}$ is Abel-summable, and that

$$(I + c_1\Theta + c_2\Theta^2 + \ldots + c_r\Theta^r)\Theta^\kappa\{s_n\}$$

converges. Then $\{s_n\}$ is summable (H,κ).

The need for 'the condition (γ)' arose from the fact that Jakimovski and Parameswaran had obtained Theorem 1 only for the case in which $\operatorname{re}\lambda \neq 0$ (the case $\operatorname{re}\lambda = 0$ corresponding to $\operatorname{re}\mu = 1$). Thus Theorem 1 of the present paper enables Theorem A to be completed by the removal of this restriction.‡

2. It has been shown by Ramanujan that there is a close relation between Hausdorff summability (H,μ_n) and quasi-Hausdorff summability (H^*,μ_{n+1}). Given a sequence $\{\mu_n\}$, (H^*,μ_{n+1}) is conservative if and only if (H,μ_n) is conservative. Further, if (H,μ_n) is regular, then so is (H^*,μ_{n+1}); if $(H^*\ \mu_{n+1})$ is regular, then so is (H,μ_n), provided that μ_0 is suitably chosen.§ Now when $\mu_n = 1/(n+1)$, the (H,μ_n) transformation reduces to Θ, while the (H^*,μ_{n+1}) transformation is given by

$$\tau_n = (n+1)\sum_{\nu=n}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)}. \tag{5}$$

† Loc. cit., Theorem A′.

‡ Cf. loc. cit., Remark (ii) of p. 297.

§ We observe that (H^*,μ_{n+1}) does not involve μ_0. The results stated here are implied by the results of Ramanujan (5) (in particular, page 201), though Ramanujan does not state them in quite this form.

Thus (5) may be regarded as the quasi-Hausdorff transformation which in some sense corresponds to Θ regarded as a Hausdorff transformation. Further, if H_λ^* denotes the transformation from $\{s_n\}$ to $\{v_n\}$ given by

$$v_n = \lambda s_n + (1-\lambda)\tau_n, \tag{6}$$

H_λ^* may be regarded as corresponding to H_λ. I have recently† considered the transformation H_λ^*, and shown that it forms an exception to the general rule that the behaviour of quasi-Hausdorff transformations is usually similar to that of the corresponding Hausdorff transformations; whereas H_λ is equivalent to convergence if and only if $\operatorname{re}\lambda > 0$, yet H_λ^* is equivalent to convergence if and only if either $\operatorname{re}\lambda < 0$, or $\lambda \neq 0$, $\operatorname{re}\alpha \geqslant 1$. However, H_λ^* does satisfy a theorem analogous to Theorem 1. We shall, in fact, prove the theorem:

THEOREM 2. *Suppose that $\lambda \neq 0$. Suppose that $\{s_n\}$ is summable H_λ^*, and that it is also Abel-summable. Then $\{s_n\}$ is convergent.*

Finally, the analogues of Theorems 1 and 2 hold with sequences replaced by functions. Let $f(x)$ be any function which is Lebesgue-integrable in any finite interval contained in $x \geqslant 0$. For $x > 0$, we write

$$g(x) = \frac{1}{x}\int_0^x f(t)\,dt, \tag{7}$$

$$h(x) = x\int_x^\infty \frac{f(t)}{t^2}\,dt, \tag{8}$$

and, for $s > 0$, we write

$$A(s) = s\int_0^\infty f(x)\,e^{-sx}\,dx, \tag{9}$$

provided, in the case of (8) and (9), that the integrals converge at infinity. Taking λ as a fixed complex number, we write

$$\theta(x) = \lambda f(x) + (1-\lambda)g(x), \tag{10}$$

$$\phi(x) = \lambda f(x) + (1-\lambda)h(x). \tag{11}$$

Then we have the theorems:

THEOREM 3. *Suppose that $\lambda \neq 0$. Suppose that $\theta(x) \to a$ as $x \to \infty$, and that $A(s) \to b$ as $s \to 0+$. Then $a = b$, and $f(x) \to a$ as $x \to \infty$.*

THEOREM 4. (i) *The proposition: 'If‡ $\phi(x) \to a$ as $x \to \infty$, then $f(x) \to a$ as $x \to \infty$' is true if and only if either* $\operatorname{re}\lambda < 0$ *or* $\lambda \neq 0$, $\operatorname{re}\alpha \geqslant 1$.

† Kuttner (4).

‡ The assertion that $\phi(x) \to a$ as $x \to \infty$ is to be taken to include the assertion that the integral on the right of (8) converges.

(ii) *Suppose that* $\lambda \neq 0$. *Suppose that* $\phi(x) \to a$ *as* $x \to \infty$, *and that* $A(s) \to b$ *as* $s \to 0+$. *Then* $a = b$, *and* $f(x) \to a$ *as* $x \to \infty$.

As the function analogue of H_λ^* does not appear to have been considered hitherto, I have, for the sake of completeness, given in Theorem 4 the analogue of the main result of Kuttner (**4**) in addition to the analogue of Theorem 2.

3. Throughout the rest of the paper, I use A to denote a non-zero constant depending only on λ; it may be different at each occurrence. We write

$$\psi(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n \tag{12}$$

whenever the series converges.

In proving Theorem 1, we may clearly suppose, without loss of generality, that $\{s_n\}$ is summable H_λ to 0; in other words, that $u_n \to 0$ as $n \to \infty$. Hence the case in which $\operatorname{re}\lambda = 0$ (which is the only case we need consider) is included in the more general theorem:

THEOREM 1*. *Suppose that* $\operatorname{re}\lambda = 0, \lambda \neq 0$, *and that* $u_n \to 0$ *as* $n \to \infty$. *Then* (12) *converges for* $0 < x < 1$, *and*

$$\psi(1-1/n) - R_n \sigma_n \to 0 \tag{13}$$

as $n \to \infty$, *where* R_n *is a certain function of* n *which tends to* A *as* $n \to \infty$.

In a similar way, the special cases of Theorems 2, 3, 4 (ii) in which $\operatorname{re}\lambda = 0$ are included respectively in the following theorems:

THEOREM 2*. *Suppose that* $\operatorname{re}\lambda = 0, \lambda \neq 0$, *and that* $v_n \to 0$ *as* $n \to \infty$. *Then* (12) *converges for* $0 < x < 1$, *and*

$$\psi(1-1/n) - K_n \tau_n \to 0 \tag{14}$$

as $n \to \infty$, *where* $K_n \to A$ *as* $n \to \infty$.

THEOREM 3*. *Suppose that* $\operatorname{re}\lambda = 0$, $\lambda \neq 0$, *and that* $\theta(x) \to 0$ *as* $x \to \infty$. *Then* (9) *converges for* $s > 0$, *and*

$$A(s) - Ag(1/s) \to 0 \tag{15}$$

as $s \to 0+$.

THEOREM 4*. *Suppose that* $\operatorname{re}\lambda = 0$, $\lambda \neq 0$, *and that* $\phi(x) \to 0$ *as* $x \to \infty$. *Then* (9) *converges for* $s > 0$, *and*

$$A(s) - Ah(1/s) \to 0 \tag{16}$$

as $s \to 0+$.

4. We now come to the proof of Theorem 1*. We write

$$r_n = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}.$$

Then, since $$s_n = -n\sigma_{n-1}+(n+1)\sigma_n,$$

it follows from (1) that

$$\frac{r_n u_n}{1+n\lambda} = -r_{n-1}\sigma_{n-1}+r_n\sigma_n.$$

Hence† $$\sigma_n = \frac{1}{r_n}\sum_{\nu=0}^{n}\frac{r_\nu u_\nu}{1+\nu\lambda}. \tag{17}$$

Now we are given that $u_n \to 0$ as $n \to \infty$, and we note that, since $\operatorname{re}\alpha = 0$, r_n and $1/r_n$ are bounded. It therefore follows from (17) that, for large n, $\sigma_n = o(\log n)$. This establishes the convergence of (12). Further, we have, for $0 < x < 1$,

$$\begin{aligned}\psi(x) &= (1-x)^2\sum_{n=0}^{\infty}(n+1)\sigma_n x^n \\ &= (1-x)^2\sum_{n=0}^{\infty}\frac{(n+1)x^n}{r_n}\sum_{\nu=0}^{n}\frac{r_\nu u_\nu}{1+\nu\lambda} = \sum_{\nu=0}^{\infty}\frac{r_\nu u_\nu}{1+\nu\lambda}R(\nu,x),\end{aligned} \tag{18}$$

where $$R(\nu,x) = (1-x)^2\sum_{n=\nu}^{\infty}\frac{(n+1)x^n}{r_n}, \tag{19}$$

the inversion in the order of summation being justified by absolute convergence. We next note that, since $1/r_n$ is bounded, we have for small positive $1-x$, uniformly in ν,

$$\begin{aligned}R(\nu,x)-R(0,x) &= -(1-x)^2\sum_{n=0}^{\nu-1}\frac{(n+1)x^n}{r_n} \\ &= O\Big\{(1-x)^2\sum_{n=0}^{\nu-1}(n+1)\Big\} = O\{(1-x)^2\nu^2\}.\end{aligned} \tag{20}$$

Also, uniformly in ν for $\nu \geqslant 1/(1-x)$,

$$\begin{aligned}R(\nu,x) &= O\Big\{(1-x)^2\sum_{n=\nu}^{\infty}(n+1)x^n\Big\} \\ &= O\{[(1-x)(\nu+1)+x]x^\nu\} = O\{(1-x)\nu x^\nu\}.\end{aligned} \tag{21}$$

Using (20) for $\nu \leqslant n$ and (21) for $\nu > n$, and using again the boundedness of r_ν, we deduce from (18) that, for large n,

$$\begin{aligned}\psi(1-1/n) = R(0,1-1/n)\sum_{\nu=0}^{n}\frac{r_\nu u_\nu}{1+\nu\lambda} + \\ +O\Big\{\frac{1}{n^2}\sum_{\nu=0}^{n}\nu|u_\nu|\Big\}+O\Big\{\frac{1}{n}\sum_{\nu=n+1}^{\infty}(1-1/n)^\nu|u_\nu|\Big\}.\end{aligned}$$

† Equation (17) is obtained (with a different notation) in the course of the proof of Hardy (**2**), Theorem 52.

Since $u_\nu = o(1)$, it follows with the aid of (17) that

$$\psi(1-1/n) = R_n \sigma_n + o(1),$$

where
$$R_n = r_n R(0, 1-1/n). \tag{22}$$

Now, for $m \geqslant 0$,

$$\frac{m+1}{r_m} = \frac{(m+1)!}{(\alpha+1)(\alpha+2)\ldots(\alpha+m)} = A(m+1)^{1-\alpha} + O(1)$$

$$= A\frac{(2-\alpha)(3-\alpha)\ldots(m+1-\alpha)}{m!} + O(1).$$

Hence, by (19), we have, for small positive $1-x$,

$$R(0,x) = A(1-x)^2 \sum_{m=0}^{\infty} \frac{(2-\alpha)(3-\alpha)\ldots(m+1-\alpha)}{m!} x^m + O\Big\{(1-x)^2 \sum_{m=0}^{\infty} x^m\Big\}$$

$$= A(1-x)^\alpha + O(1-x). \tag{23}$$

Since $r_n \sim An^\alpha$, it follows from (22) and (23) that $R_n \to A$ as $n \to \infty$. This completes the proof of Theorem 1*.

5. The cases of Theorem 2 in which $\operatorname{re}\lambda < 0$ or $\lambda \neq 0$, $\operatorname{re}\alpha \geqslant 1$ are included in Theorem 1 of Kuttner (**4**); in these cases the hypothesis of Abel summability is redundant. Now suppose that $0 < \operatorname{re}\alpha < 1$. We write

$$k_n = \frac{n!}{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}. \tag{24}$$

Rewriting equation (14) of Kuttner (**4**) in the notation of the present paper,† we have

$$\frac{k_n v_n}{\lambda n + \lambda + 1} = k_n \tau_n - k_{n+1}\tau_{n+1}, \tag{25}$$

whence it follows that, whenever the series on the right converges,

$$\tau_n = \frac{C}{k_n} + \frac{1}{k_n}\sum_{\nu=n}^{\infty} \frac{k_\nu v_\nu}{\lambda\nu + \lambda + 1}, \tag{26}$$

where C is a constant. Now suppose that $\{s_n\}$ is summable H_λ^* to 0, so that $v_n \to 0$ as $n \to \infty$. Since $k_n \sim An^{-\alpha}$, we deduce that (26) is valid, and that

$$\tau_n = C/k_n + o(1).$$

Hence, by (6),
$$s_n = (1-\alpha)C/k_n + o(1).$$

If, now, we suppose further that $\{s_n\}$ is Abel-summable (or, more generally, that the Abel means of $\{s_n\}$ are bounded) it follows that $C = 0$. This establishes the case of Theorem 2 in which $0 < \operatorname{re}\alpha < 1$.

† We observe that the 'k_n' of (**4**) (with $N = 0$) is $n+1$ times the present 'k_n'.

We now come to the proof of Theorem 2* (which includes the only case of Theorem 2 still outstanding). Since any alteration in s_0 will not affect either the hypotheses or the conclusion of the theorem, we may suppose s_0 changed (if necessary) in such a way as to make $\tau_0 = 0$; it then follows from (25) that

$$\tau_n = -\frac{1}{k_n}\sum_{\nu=0}^{n-1}\frac{k_\nu v_\nu}{\lambda\nu+\lambda+1}. \tag{27}$$

Now (6) can be written in the form

$$s_n = \alpha v_n + (1-\alpha)\tau_n,$$

whence
$$\psi(x) = \alpha\psi_1(x) + (1-\alpha)\psi_2(x), \tag{28}$$

where
$$\psi_1(x) = (1-x)\sum_{n=0}^{\infty} v_n x^n \to 0 \tag{29}$$

as $x \to 1-$, and where

$$\psi_2(x) = (1-x)\sum_{n=0}^{\infty}\tau_n x^n.$$

It follows from (27) that

$$\psi_2(x) = -(1-x)\sum_{n=0}^{\infty}\frac{x^n}{k_n}\sum_{\nu=0}^{n-1}\frac{k_\nu v_\nu}{\lambda\nu+\lambda+1} = -\sum_{\nu=0}^{\infty}\frac{k_\nu v_\nu}{\lambda\nu+\lambda+1}K(\nu,x), \tag{30}$$

where
$$K(\nu,x) = (1-x)\sum_{n=\nu+1}^{\infty}\frac{x^n}{k_n}. \tag{31}$$

Again, since $1/k_n$ is bounded, we have

$$K(\nu,x) - K(0,x) = -(1-x)\sum_{n=1}^{\nu}\frac{x^n}{k_n} = O\{\nu(1-x)\},$$

and also
$$K(\nu,x) = O(x^\nu).$$

It therefore follows from (30) that

$$\psi_2(1-1/n) = -K(0,1-1/n)\sum_{\nu=0}^{n}\frac{k_\nu v_\nu}{\lambda\nu+\lambda+1} + \\ + O\left\{\frac{1}{n}\sum_{\nu=0}^{n}|v_\nu|\right\} + O\left\{\sum_{\nu=n+1}^{\infty}\frac{|v_\nu|}{\nu}\left[1-\frac{1}{n}\right]^\nu\right\}.$$

Hence, again using (27), we have

$$\psi_2(1-1/n) = K_n\tau_n + o(1), \tag{32}$$

where
$$K_n = k_n K(0,1-1/n).$$

Clearly, (14) follows from (28), (29), and (32). Also, by (24) and (31),

$$K(0,x) = (1-x)^{-\alpha} - (1-x).$$

Hence $K_n \to A$ as $n \to \infty$, and the proof of the theorem is completed.

6. As with the analogous Theorem 1, those cases of Theorem 3 in which $\operatorname{re}\lambda \neq 0$ follow from a known theorem.† It is therefore enough to consider Theorem 3*. The hypothesis and conclusions of the theorem are unaffected if we alter $f(x)$ in any finite interval; we may therefore suppose that $f(x)$, and hence also $g(x)$ and $\theta(x)$, vanish identically in some right-hand neighbourhood of the origin. It then follows from (7) and (10) that

$$g(x) = \frac{\alpha}{x^\alpha} \int_0^x t^{\alpha-1}\theta(t)\,dt. \tag{33}$$

Hence the hypothesis that $\theta(t) \to 0$ as $t \to \infty$ implies that‡ $g(x) = o(\log x)$ for large x, and hence, by (10), that $f(x) = o(\log x)$. This establishes the convergence of (9). Also, integrating by parts, we have

$$A(s) = s^2 \int_0^\infty xg(x)e^{-sx}\,dx = \alpha s^2 \int_0^\infty x^{1-\alpha}e^{-sx}\,dx \int_0^x t^{\alpha-1}\theta(t)\,dt,$$

by (33). Hence, the inversion in the order of integration being justified by absolute convergence,

$$A(s) = \alpha s^2 \int_0^\infty t^{\alpha-1}\theta(t)\,dt \int_t^\infty x^{1-\alpha}e^{-sx}\,dx = \alpha s^\alpha \int_0^\infty t^{\alpha-1}\theta(t)\psi(st)\,dt, \tag{34}$$

where

$$\psi(\xi) = \int_\xi^\infty v^{1-\alpha}e^{-v}\,dv.$$

Using the results that, for small ξ,

$$\psi(\xi) - \psi(0) = -\int_0^\xi v^{1-\alpha}e^{-v}\,dv = O(\xi^2),$$

and that, for large ξ, $\quad \psi(\xi) = O(\xi e^{-\xi})$,

we can now complete the proof by an argument analogous to that used for Theorem 1*.

7. We now consider the negative part of Theorem 4 (i). We have to show that, if $\lambda \neq 0$, $0 \leqslant \operatorname{re}\alpha < 1$,§ the hypothesis that $\phi(x) \to a$ as $x \to \infty$ does not imply that $f(x) \to a$ as $x \to \infty$. For this, we take

$$f(x) = x^\alpha \quad (x \geqslant 1).$$

So long as $\operatorname{re}\alpha < 1$, (8) converges, and we have, for $x \geqslant 1$,

$$h(x) = x^\alpha/(1-\alpha),$$

† Hardy (**1**), Theorem 1. Hardy supposes that $f(x)$ is continuous, but this restriction is not essential.

‡ This result is given in Hardy (**1**).

§ The case $\lambda = 0$ is familiar, and may therefore be omitted.

and hence $$\phi(x) = 0.$$

This gives the required result.

In dealing with the rest of Theorem 4 (and with Theorem 4*) we may suppose, as in the case of Theorem 3*, that $f(x)$ vanishes identically in some right-hand neighbourhood of the origin. We may further alter $f(x)$ (if necessary) in such a way as to arrange that

$$\int_0^\infty \frac{f(t)}{t^2}\,dt = 0.$$

Then $h(x)$ and $\phi(x)$ will also vanish identically in some right-hand neighbourhood of the origin. It therefore follows from (8) and (11) that

$$h(x) = -\alpha x^\alpha \int_0^x t^{-\alpha-1}\phi(t)\,dt. \tag{35}$$

Suppose now that $$\phi(x) \to 0 \tag{36}$$

as $x \to \infty$. If $\operatorname{re}\alpha < 0$, it follows at once from (35) that $h(x) \to 0$ as $x \to \infty$. If $\operatorname{re}\alpha \geqslant 1$, then, since the convergence of (8) alone implies that $h(x) = o(x)$, and thus that

$$x^{-\alpha}h(x) \to 0$$

as $x \to \infty$, we deduce from (35) that

$$h(x) = \alpha x^\alpha \int_x^\infty t^{-\alpha-1}\phi(t)\,dt. \tag{37}$$

It follows from (36) and (37) that $h(x) \to 0$ as $x \to \infty$, and this completes the proof of Theorem 4 (i).

Those cases of Theorem 4 (ii) in which $\operatorname{re}\lambda < 0$ or $\operatorname{re}\alpha \geqslant 1$ are included in Theorem 4 (i). Now it follows from (35) that (for any $\lambda \neq 0$)

$$h(x) = Kx^\alpha + \alpha x^\alpha \int_x^\infty t^{-\alpha-1}\phi(t)\,dt, \tag{38}$$

where K is a constant, whenever the integral on the right converges. Suppose now that $0 < \operatorname{re}\alpha < 1$. If (36) holds, we deduce that (38) is valid, and hence that, for large x,

$$h(x) = Kx^\alpha + o(1).$$

If we suppose further that $A(s)$ tends to a limit as $s \to 0+$ (or, more generally, that $A(s)$ is bounded for $s > 0$), it follows that $K = 0$, which gives this case of the theorem.

We now come to the proof of Theorem 4*, which includes the only

remaining case of Theorem 4 (ii). We have, by (11),

$$A(s) = \alpha A_1(s) + (1-\alpha)A_2(s),$$

where
$$A_1(s) = s\int_0^\infty \phi(x)e^{-sx}\,dx \to 0$$

as $s \to 0+$, by (36), and where

$$A_2(s) = s\int_0^\infty h(x)e^{-sx}\,dx = -\alpha s\int_0^\infty x^\alpha e^{-sx}\,dx\int_0^x t^{-\alpha-1}\phi(t)\,dt$$

$$= -\alpha s\int_0^\infty t^{-\alpha-1}\phi(t)\,dt\int_t^\infty x^\alpha e^{-sx}\,dx = -\alpha s^{-\alpha}\int_0^\infty t^{-\alpha-1}\phi(t)\chi(st)\,dt,$$

where
$$\chi(\xi) = \int_\xi^\infty v^\alpha e^{-v}\,dv.$$

Since, for small ξ,
$$\chi(\xi) - \chi(0) = O(\xi),$$

and, for large ξ,
$$\chi(\xi) = O(e^{-\xi}),$$

we can now complete the proof by an argument analogous to that used for Theorem 2*.

REFERENCES

1. G. H. Hardy, 'Generalizations of a limit theory of Mr. Mercer', *Quart. J. of Math.* 43 (1912) 143–50.
2. —— *Divergent series* (Oxford, 1949).
3. A. Jakimovski and M. R. Parameswaran, 'Generalized Tauberian theorems for summability-(A)', *Quart. J. of Math.* (Oxford) (2) 9 (1958) 290–8.
4. B. Kuttner, 'On a certain quasi-Hausdorff transformation', *J. London Math. Soc.* 34 (1959) 401–5.
5. M. S. Ramanujan, 'On Hausdorff and quasi-Hausdorff methods of summability', *Quart. J. of Math.* (Oxford) (2) 8 (1957) 197–213.

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